Lecture 3

Combinatorics, Bernoulli Trials, Poisson Limit Theorem

Combinatorics is an area of mathematics primarily concerned with counting, and certain properties of finite structures.

It is closely related to many other areas of mathematics and has many applications ranging from logic to statistical physics, from evolutionary biology to computer science, etc.

Gottfried Wilhelm von Leibniz
1646-1716

In combinatorial analysis (combinatorics), we basically deal with the following two questions:

- ❑ In how many ways can a set of elements be *sequenced* (*arranged*)?
- ❑ In how many ways can a subset be *selected* from a set of elements?

Since the counting of all possible combinations can become quite complicated, we are going to present a systematic discussion of the methods required for the most important counting problems that occur in the applications of the probability theory.

Such counting problems are called combinatorial problems, because we count the numbers of ways in which different possible outcomes can be combined.

In combinatorics, we distinguish between *ordered* and *unordered* sets. In an ordered set, the order plays a role, whereas in an unordered set, it does not.

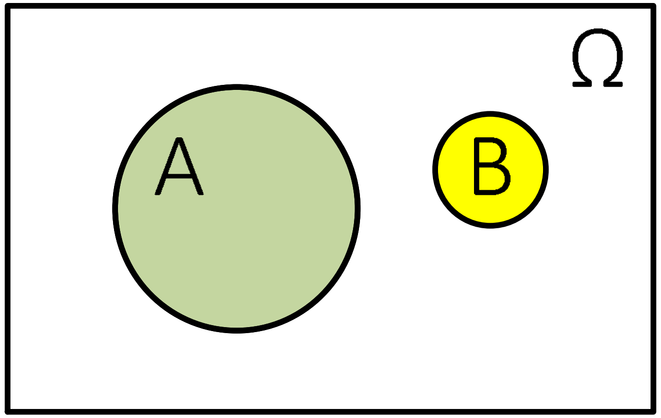
For instance, the list of all ordered subsets of size two of $\{1,2,3\}$ consists of $(1,2), (2,1), (1,3), (3,1), (2,3)$ and $(3,2)$; the list of unordered subsets of size two consists of $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$.

The set $\{2,1\}$ is the same as $\{1,2\}$ and therefore not listed separately.

The first question we ask is: What do our basic set operations do to the numbers of elements of the sets involved?

In other words if we let $|X|$ denote the number of elements (cardinality) of the set X, then how are $|A|$, $|B|$, $|A \cup B|$, $|A \cap B|$, $|A^c|$, $|A \backslash \mathrm{B}|$, etc. related to each other?

Addition Principle



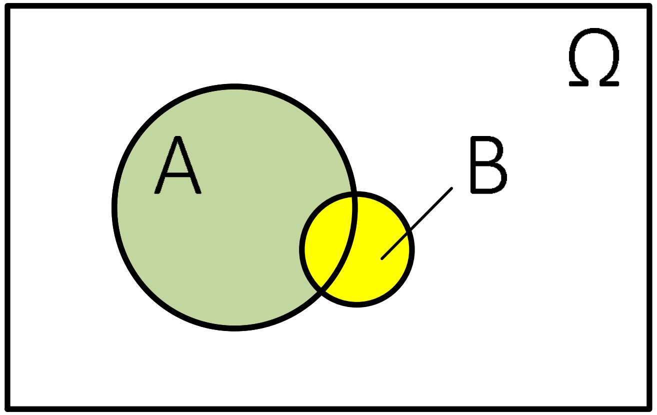
If $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$

If $A_1, A_2, \ldots, A_k$ are k disjoint sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_k| = |A_1| + |A_2| + \cdots + |A_k|$$

Addition Principle

For any two sets A and B



$$|A \cup B| = |A| + |B| - |A \cap B|.$$

In a survey, 100 people are asked whether they drink or smoke or do both or neither.

The results are 60 drink, 30 smoke, 20 do both, and 30 do neither.

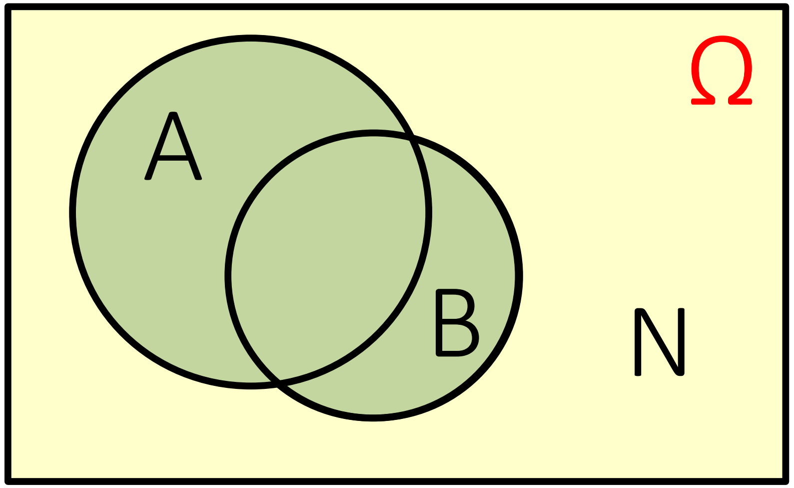
Are these numbers compatible with each other?

If we let A denote the set of drinkers, B the set of smokers, N the set of those who do neither, and Ω the set of all those surveyed, then the data translate to

$$|A| = 60 \qquad |B| = 30 \qquad |A \cap B| = 20$$

$$|N| = 30 \qquad |\Omega| = 100$$

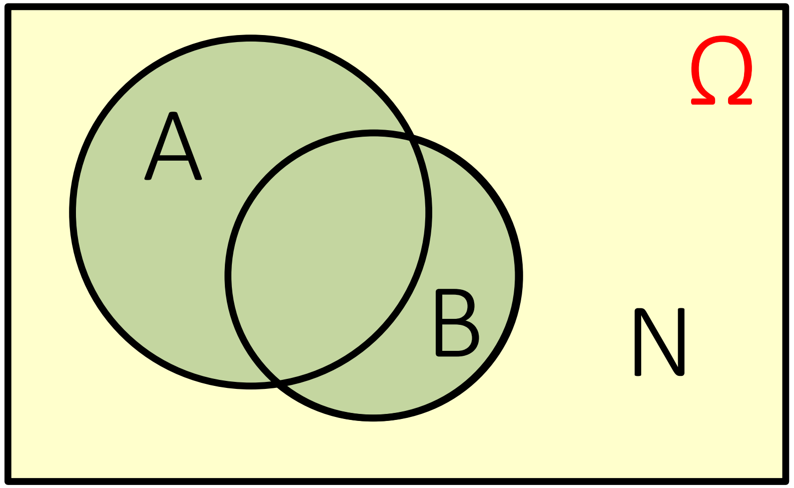
Also, $A \cup B$ and N are disjoint, and $A \cup B \cup N = \Omega$.



So we must have

$|A \cup B| + |N| = |\Omega|$, that is,

$|A \cup B| + 30 = 100.$



According to addition principle

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Therefore in our case

$$|A \cup B| = 60 + 30 - 20 = 70,$$

and $|A \cup B| + 30$ is indeed 100, which shows that the data are compatible.

Inclusion-Exclusion Theorem

For any positive integer N and arbitrary sets $A_1, A_2, \dots, A_N$

$$\left|\bigcup_{i=1}^{N} A_i\right| = \sum_{1\le i\le N} |A_i| - \sum_{1\le i<j\le N} \left|A_i \cap A_j\right| +$$

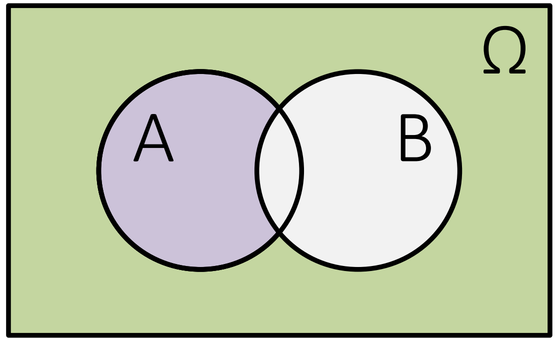
$$+ \sum_{1\le i<j<k\le N} \left|A_i \cap A_j \cap A_k\right| - \cdots + (-1)^{N-1} \cdot \left|A_i \cap A_j \cap \cdots \cap A_N\right|$$

See proof in *Géza Schay. Introduction to Probability…, p. 28*

Subtraction Principle

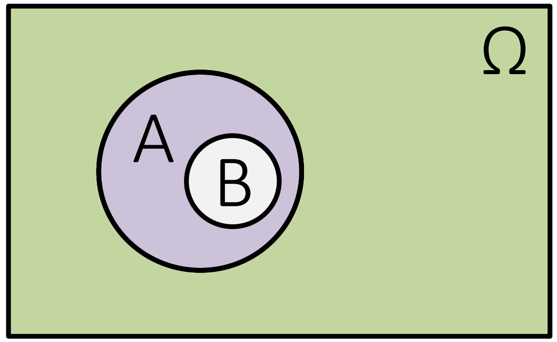
For any two sets A and B

$$|A \backslash \mathrm{B}| = |A| - |A \cap B|$$



Specifically,

$$|A \backslash \mathrm{B}| = |A| - |B| \leftrightarrow B \subset A$$



Also,

$$|A^C| = |\Omega \backslash \mathrm{A}| = |\Omega| - |A|$$

How many positive integers ≤ 1000 are there that are not divisible by 6, 7, and 8?

$$\Omega = \{1, 2, \ldots, 1000\}$$

$$A = \{multiples\ of\ 6\ in\ \Omega\} \qquad B = \{multiples\ of\ 7\ in\ \Omega\}$$

$$C = \{multiples\ of\ 8\ in\ \Omega\}$$

Then $|\Omega| = 1000$

$|A| = \lfloor 1000/6 \rfloor = 166$

$|B| = \lfloor 1000/7 \rfloor = 142$

$|C| = \lfloor 1000/8 \rfloor = 125$

$|A \cap B| = \lfloor 1000/42 \rfloor = 23$

$|B \cap C| = \lfloor 1000/56 \rfloor = 17$

$|A \cap C| = \lfloor 1000/24 \rfloor = 41$

least common multiple

$|A \cap B \cap C| = \lfloor 1000/168 \rfloor = 5$

By definition, the set of integers not divisible by 6, 7 and 8 is a complement set of the union $A \cup B \cup C$.

Then

$$|(A \cup B \cup C)^C| = 1000 - \\ -166 - 142 - 125 + \\ +23 + 41 + 17 - \\ -5 = 643$$

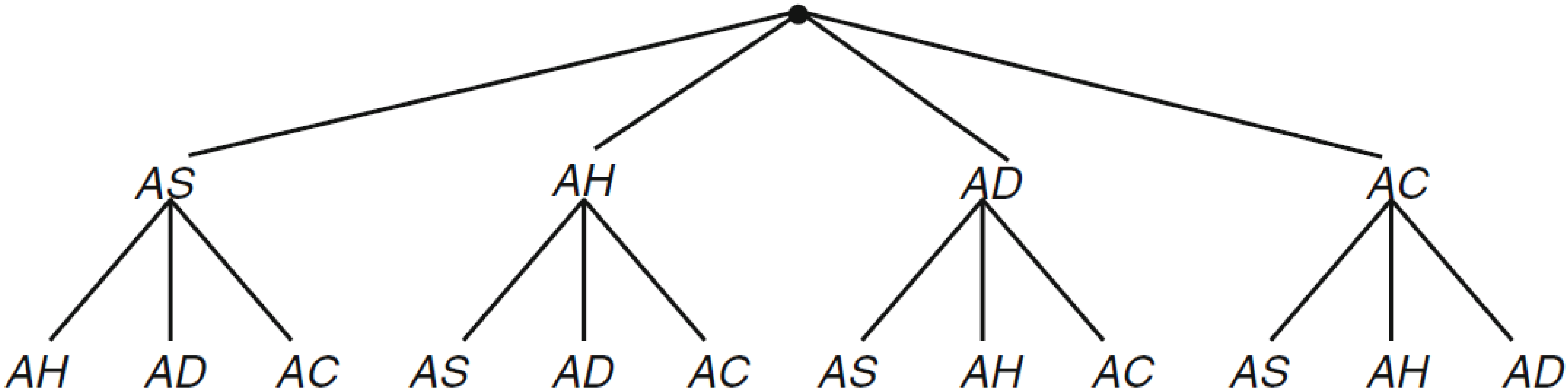
So far we worked with fixed sample spaces and counted the number of points in single events.

Now we are going to consider the construction of new sample spaces and events from previously given ones and count the number of possibilities in the new sets.

For example, we draw two cards from a deck and want to find the number of ways in which the two drawings both result in Aces.

The best way to approach such problems is by drawing a so-called *tree diagram*. In such diagrams we first list the possible outcomes of the first step and then draw lines from each of those to the elements in a list of the possible outcomes that can occur in the second step depending on the outcome in the first step.

Let us illustrate the possible ways of successively drawing two Aces from a deck of cards.

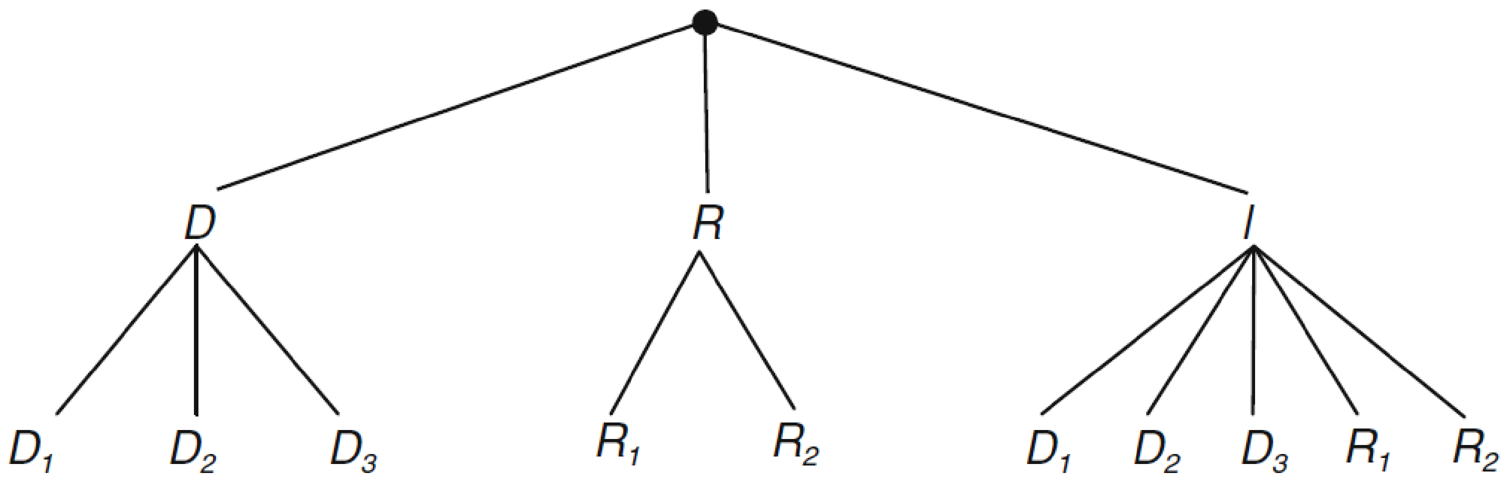


In the first step, we can obtain AS, AH, AD, AC, but in the second step, we can only draw an Ace that has not been drawn before.

Before primary elections, voters are polled about their preferences in a certain state. There are two Republican candidates R_1 and R_2 and three Democratic candidates D_1, D_2, and D_3.

The voters are first asked whether they are registered Republicans (R), Democrats (D), or independents (I) and, second, which candidate they prefer.

The independents are allowed to vote in either primary, so in effect they can choose any of the five candidates.



The branches correspond to mutually exclusive events in the 10-element sample space:
$\{DD_1, DD_2, DD_3, RR_1, RR_2, ID_1, ID_2, ID_3, IR_1, IR_2\}$.

This is the new sample space built up from the simpler ones $\{D, R, I\}$, $\{D_1, D_2, D_3\}$, and $\{R_1, R_2\}$.

The Multiplication Principle

If an experiment is performed in m steps, and there are n_1 choices in the first step, and for each of those there are n_2 choices in the second step, and so on, with n_m choices in the last step for each of the previous choices, then the number of possible outcomes, for all the steps together, is given by the product

$$n_1 \cdot n_2 \cdot \cdots \cdot n_m = \prod_{i=1}^{m} n_i$$

Suppose three cards are drawn from a regular deck of 52 cards.

What is the number of ways they can be drawn

- if we return each card into the deck before the next one is drawn?
- if we do not return cards?

For each case we have a 3-step experiment ($m = 3$).

If cards are being returned into the deck, $n_1 = n_2 = n_3 = 52$.
Then, we can draw 3 cards in $52^3 = 140608$ ways.

If cards are not being returned, $n_1 = 52$, $n_2 = 51$, $n_3 = 50$, and we can draw 3 cards in $52 \cdot 51 \cdot 50 = 132600$ ways.

Results of the last example represent a concept known as *permutation* of elements of the set:

- *with repetitions* in the former case,
- and *without repetitions* in the latter.

Any arrangement of things in a *ordered* row is called a *permutation* of those things.

Permutations

Suppose now that we have n distinct objects and that we take, at random and without replacement, k objects among them.

The number of possible arrangements is given by

$$P_k^n = n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

Sometimes, notations ${}_nP_k$ or even P_n^k are used.

Permutations

P_k^n specifies *partial permutations* or k-*permutation* on n items.

If all n items need to be arranged, then the number of permutations is

$$P_n^n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!,$$

since $0! = 1$ by the definition of factorial.

Combinations

In many problems it is unnatural to concern ourselves with the order in which things are selected, and we want to count only the number of different possible selections without regard to order.

The number of possible *unordered* selections of k different things out of n different ones is called a combination of the given things.

Combinations

If the order of the objects is not important, then the number of ways to take, at random and without replacement, k objects among n distinct objects is given by

$$C_k^n = \binom{n}{k} = \frac{P_k^n}{k!} = \frac{n!}{k! \cdot (n-k)!}.$$

C_k^n is often pronounced "*n* choose *k*" and represent the number of k-combinations out of n items.

In a class there are 30 men and 20 women. In how many ways can a committee of two men and two women be chosen?

We have to choose 2 men out of 30 and 2 women out of 20. These choices can be done in C_2^{30} and C_2^{20} ways, respectively.

By the multiplication principle, the whole committee can be selected in

$$C_2^{30} \cdot C_2^{20} = \frac{30!}{2!\cdot 28!} \cdot \frac{20!}{2!\cdot 18!} = \frac{30\cdot 29}{2} \cdot \frac{20\cdot 19}{2} = 82650 \text{ ways.}$$

Permutations with Repetitions

We have discussed permutations of objects different from each other. Now, we consider permutations of objects, some of which may be identical or which amounts to the same thing of different objects that may be repeated in the permutations.

Permutations with Repetitions

In general, if we have k different objects and we consider permutations of length n, with the first object occurring n_1 times, the second n_2 times, and so on, with the kth object occurring n_k times,
then we must have $n_1 + n_2 + \cdots + n_k = n$, and the number of such permutations is

$$\frac{n!}{n_1!\, n_2! \ldots n_k!}.$$

Permutations with Repetitions

This quantity is called a *multinomial coefficient* and is sometimes denoted by

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\, n_2! \ldots n_k!}.$$

How many seven-letter words can be made up of two *a*'s, two *b*'s, and three *c*'s?

Here $n = 7$, $k = 3$, $n_1 = n_2 = 2$, and $n_3 = 3$.

Thus the answer is

$$\binom{7}{2,2,3} = \frac{7!}{2!\,2!\,3!} = 210.$$

Bernoulli trials

Consider an experiment that consists of n identical and statistically independent sub-experiments called trials. In each trial we have:

1. Two possible outcomes, which we call "success" and "failure";

2. The probability of success is the same number p in each trial, while the probability of failure is $q = 1 - p$.

Such trials are called Bernoulli trials.

For example, tossing a coin or throwing a die repeatedly or selecting a person from a given population with replacement and observing whether he or she has a certain trait are such trials.

Jacob Bernoulli
1654-1705

Often, the sequence of Bernoulli trials is called a binomial experiment.

We ask for the probability $b(k;n,p)$ of obtaining exactly k successes in the n trials.

By the assumed independence, the probability of having k successes and $n-k$ failures in any fixed order is $p^k q^{n-k}$, and since the k successes and $n-k$ failures can be ordered in C_k^n mutually exclusive ways

$$b(k;n,p) = C_k^n p^k q^{n-k}.$$

In an airport, five radars are in operation and each radar has a $p = 0.9$ probability of detecting an arriving airplane. The radars operate independently of each other.

a) Calculate the probability that an arriving airplane will be detected by at least four radars.

b) Knowing that at least three radars detected a given airplane, what is the probability that the five radars detected this airplane?

c) What is the smallest number of radars that must be installed if we want an arriving airplane to be detected by at least one radar with probability 0.9995 or greater?

(a) Let X be the number of radars that successfully detect the airplane. Then, the probability that an arriving airplane will be detected by at least four radars is

$$\Pr\{X \geq 4\} = b(4; 5{,}0.9) + b(5; 5{,}0.9)$$

$$\Pr\{X \geq 4\} = C_4^5 \cdot 0.9^4 \cdot 0.1^1 + C_5^5 \cdot 0.9^5 \cdot 0.1^0$$

$$\Pr\{X \geq 4\} = 5 \cdot 0.9^4 \cdot 0.1^1 + 1 \cdot 0.9^5 \cdot 0.1^0 \cong 0.9185$$

(b) We want the conditional probability

$$\Pr\{X = 5 \mid X \geq 3\} = \frac{\Pr\{(X = 5) \cap (X \geq 3)\}}{\Pr\{X \geq 3\}}.$$

Given that $\{X = 5\} \subset \{X \geq 3\}$, the intersection of these two sets is a set $\{X = 5\}$.

$$\Pr\{X = 5 \mid X \geq 3\} = \frac{\Pr\{X = 5\}}{\Pr\{X \geq 3\}}.$$

$$\Pr\{X = 5\} = b(5; 5,0.9) = C_5^5 \cdot 0.9^5 \cdot 0.1^0 \cong 0.5905$$

$$\Pr\{X \geq 3\} = C_3^5 \cdot 0.9^3 \cdot 0.1^2 + C_4^5 \cdot 0.9^4 \cdot 0.1^1 +$$

$$+C_5^5 \cdot 0.9^5 \cdot 0.1^0 \cong 0.9914$$

$$\Pr\{X = 5 \mid X \geq 3\} \cong \frac{0.5905}{0.9914} \approx 0.596.$$

(c) We want to find the smallest n such that

$$\Pr\{X \geq 1\} \geq 0.9995.$$

It's easier to compute $1 - \Pr\{X = 0\}$:

$$\Pr\{X \geq 1\} = 1 - \Pr\{X = 0\} = 1 - C_0^n \cdot 0.9^0 \cdot 0.1^n =$$
$$= 1 - (0.1)^n \geq 0.9995;$$

$$(0.1)^n \leq 0.0005 \;\Rightarrow n \geq \log_{0.1} 0.0005 \approx 3.3 \;\Rightarrow\; n_{min} = 4$$

$$b(k;n,p) = C_k^n p^k q^{n-k}$$

Computing probabilities with Bernoulli's formula is convenient only if n is relatively small, since we must calculate $n!$ and such.

If n is very large and probability of success p is small we face even greater challenge using Bernoulli's formula.

Poisson Limit Theorem

Consider a sequence of n Bernoulli trials with success probability p and failure probability $q = 1 - p$.

If $n \to \infty$, $p \to 0$ and $np \to \lambda > 0$, then

$$b(k; n, p) \approx \frac{\lambda^k}{k!} \cdot e^{-\lambda}.$$

This result is known as Poisson Limit Theorem (PLT) or the Law of Rare Events.

Siméon Denis Poisson

1781-1840

A brewery sent a shipment of 100,000 bottles of beer to a customer. There is a 0.0001 probability that a bottle breaks during delivery.

What is the probability that exactly 4 bottles break during delivery?

We have $n = 100000$, $p = 0.0001$, and $np = \lambda = 10$.

According to PLT

$$b(4; 10^5, 10^{-4}) \approx \frac{10^4}{4!} \cdot e^{-10} \approx 0.019.$$

Textbook Assignment

Géza Schay. *Introduction to Probability…*

- Chapter 3. 25-51 pp.
- Ex. 3.5.2, 3.5.3, 3.5.7 and 3.5.10