Probability Theory

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Syllabus

54 contact hours

3 credits

Control: Graded Test

Grading Policy:

0 – 54	**fail**
55 – 64	**E**
65 – 69	**D**
70 – 79	**C**
80 – 89	**B**
90 – 100	**A**

Topics:

Combinatorics. Probability spaces and events. Random variables. Probability distributions. Law of large numbers. Central limit theorem. Basic concepts of Statistics.

Recommended Readings

Géza Schay, Introduction to Probability with Statistical Applications, (2016). Birkhäuser, 2nd ed.

F.M. Dekking et al., A Modern Introduction to Probability and Statistics, (2005). Springer London.

*The books are available at **SpringerLink**:*

https://ezproxy.ha.tpu.ru:2443/login?url=http://link.springer.com/

May require your TPU Account credentials

Further Readings

Mario Lefebvre, Basic Probability Theory with Applications, (2009). Springer Science+Business Media.

Ronald Meester, A Natural Introduction to Probability Theory, (2008). Birkhäuser, 2nd ed.

Anirban DasGupta, Fundamentals of Probability: A First Course, (2010). Springer Science+Business Media.

*Also available at **SpringerLink**.*

Lecture 1

The Algebra of Events

The Algebra of Events

Randomness: apparent lack of predictability in events.

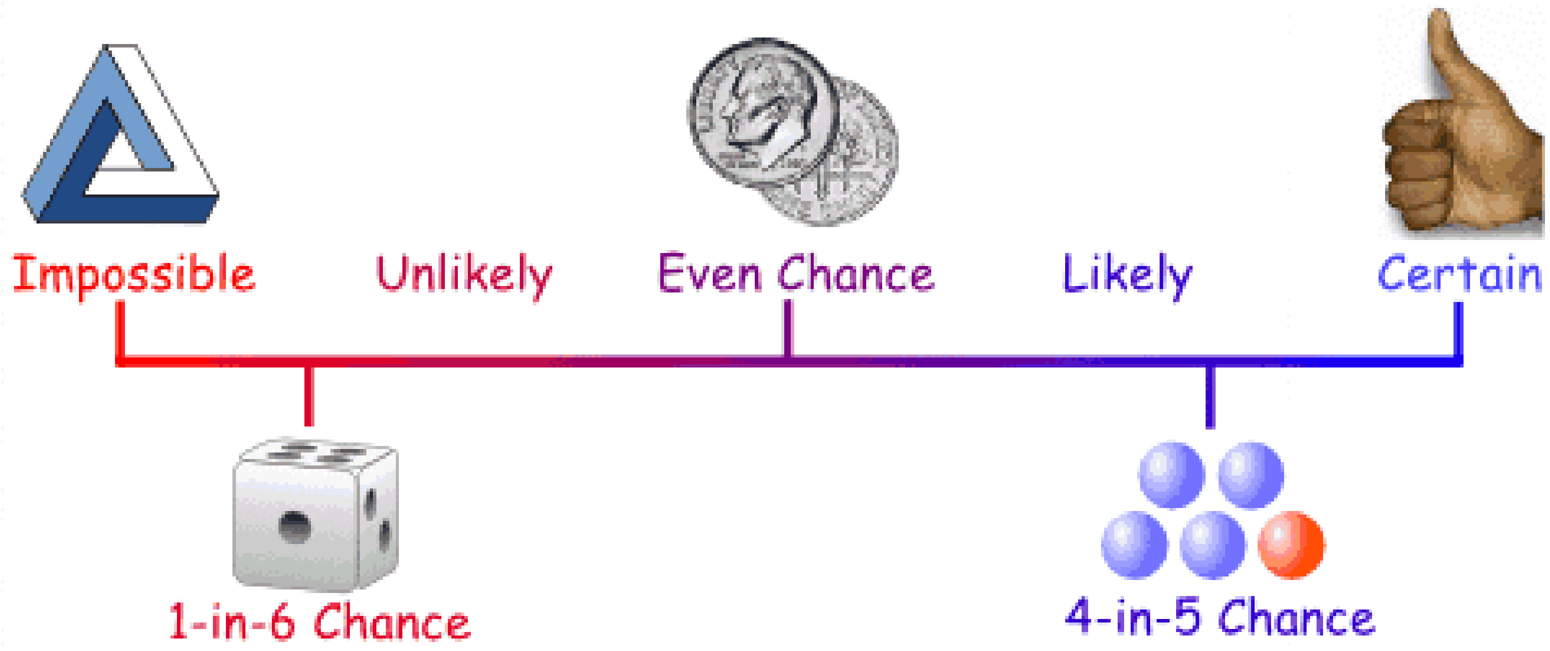
A random sequence of events does not follow an intelligible pattern.

Probability is a quantitative measure of randomness, calibrated on a scale of 0 to 1.

The Algebra of Events

Probability equals 0 : impossible event.

Probability equals 1 : certain event.



The Algebra of Events

A random experiment is an experiment that, at least theoretically, may be repeated infinitely many times independently and whose outcome cannot be predicted, for example, the roll of dice.

The Algebra of Events

Each time the experiment is repeated, an elementary outcome is obtained.

The ***set*** of **all** elementary outcomes of a random experiment is called the sample space, which is denoted by Ω.

Sample spaces may be ***discrete*** or ***continuous***.

The Algebra of Events

Discrete sample spaces (**finite**):

- the number that shows up when rolling a die

$$\Omega = \{1,2,3,4,5,6\}$$

- the outcome of a coin tossed twice

$$\Omega = \{HH, HT, TH, TT\}$$

- the number of calls received on a given day

$$\Omega = \{0,1,2,\ldots,n\}, \qquad n \neq \infty$$

Discrete sample spaces (**infinite**):

- ☐ the number of die rolls made before getting first “6”

$$\Omega = \{1,2,3, \dots\}$$

- ☐ selecting an integer number from an interval $[-10, \infty)$

$$\Omega = \{-10, -9, -8, \dots\}$$

Continuous sample spaces:

- ☐ the time needed to get the first “6” when rolling a die

$$\Omega = \{t \in \mathbb{R}: t > 0\}$$

- ☐ choosing a number at random from the interval $[0, 1]$

$$\Omega = \{x \in \mathbb{R}: 0 \leq x \leq 1\}$$

The Algebra of Events

An event is a set of elementary outcomes.

That is, it is a subset of the sample space .

$$E \subseteq \Omega$$

In particular, every elementary outcome is an event, and so is the sample space itself.

Remarks

An elementary outcome is sometimes called a *simple event*, whereas a *compound event* is made up of at least two elementary outcomes.

We should distinguish between the elementary outcome ω, which is an element of Ω, and the elementary event, which is a subset of Ω.

$\omega \in \Omega$ - outcome

$\{\omega\} \subset \Omega$ - event

The Algebra of Events

Consider the experiment that consists in rolling a die and recording the number that shows up.

Then, the sample space is $\Omega = \{1,2,3,4,5,6\}$.

We can define the events

$$A = \{1,2,4}, \qquad B = \{2,4,6\}, \qquad C = \{3,5\}, \qquad D = \{6\}$$

compound events

simple event

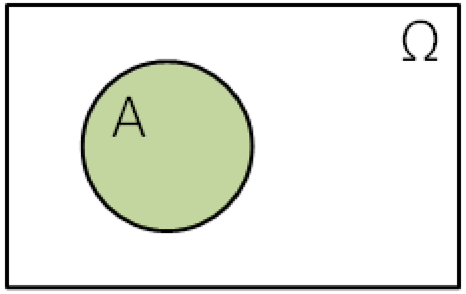
Set theory notation

Notation	Set Theory	Events
Ω	universal set	sample space, certain event
$\emptyset$	empty (null) set	impossible event
$A \subset \Omega, A \subseteq \Omega$	(proper) subset	event A happens
$A^C, \bar{A}$	complement set	complement event, “A does not happen”
$A \cap B, AB$	intersection	both A and B happen
$A \cup B$	union	at least one of A and B happens
$A \setminus B, A - B$	set difference	A happens, but B does not
$A \triangle B$	symmetric difference	exactly one of A and B happens

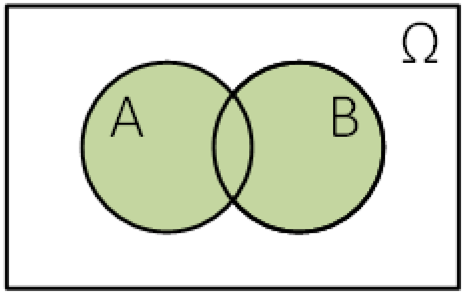
The Algebra of Events

Venn diagrams

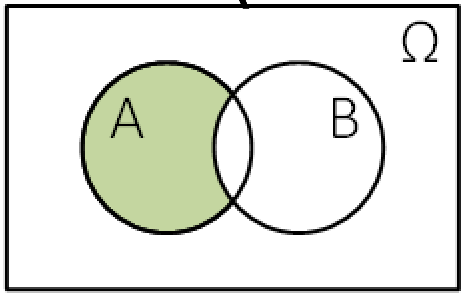
A



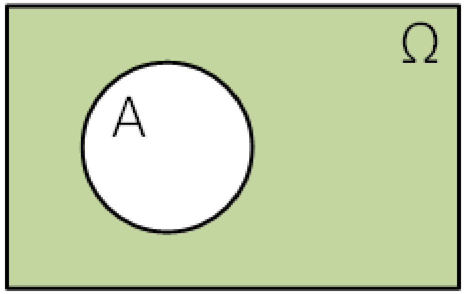
$A \cup B$



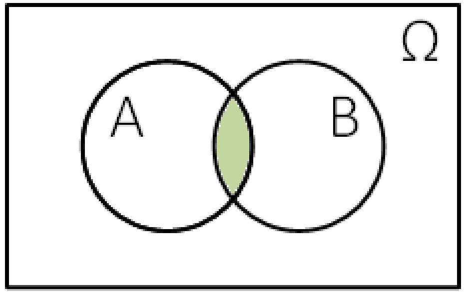
$A \setminus B$



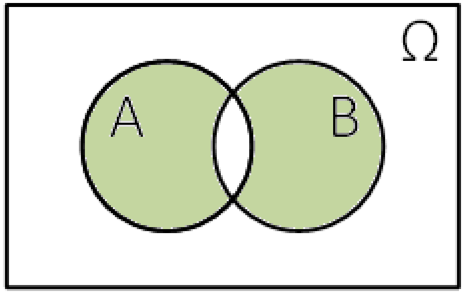
A^C



$A \cap B$



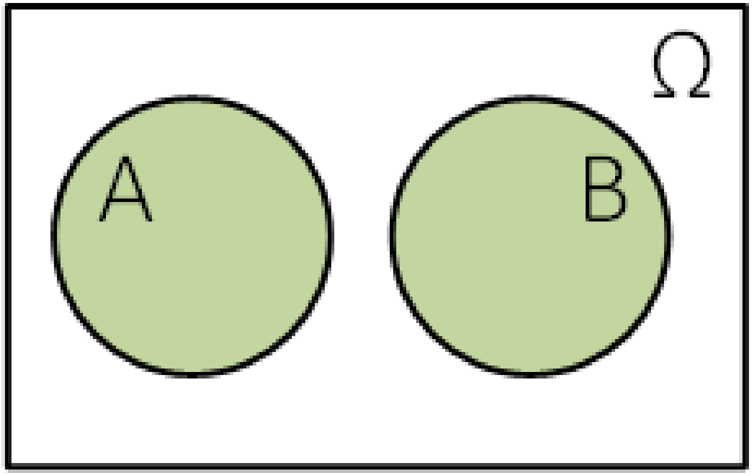
$A \triangle B$



The Algebra of Events

Two (or more) events are said to be **mutually exclusive** (*disjoint, incompatible*) if their intersection is a null set.

$$A \cap B = \emptyset$$



The Algebra of Events

Consider the experiment that consists in rolling a die and recording the number that shows up. We have defined the following events:

$$A = \{1,2,4\}, \qquad B = \{2,4,6\}, \qquad C = \{3,5\}, \qquad D = \{6\}$$

Then

$$A \cup B = \{1,2,4,6\}, \qquad C \cup D = \{3,5,6\}$$

$$A \cap B = \{2,4\}, \qquad C \cap D = \emptyset$$

$$\Omega \setminus A = A^C = \{3,5,6\}, \qquad A \setminus B = \{1\}$$

$$A \bigtriangleup B = \{1,6\}$$

The Algebra of Events

Events $A_1, \ldots, A_n$ form a partition of the sample space Ω if they are mutually exclusive:

$$A_i \cap A_j = \emptyset, i \neq j$$

and

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n = \Omega.$$

The other term for partition is a set of *exhaustive events*.

The Algebra of Events

Consider the experiment that consists in rolling a die and recording the number that shows up. We have defined the following events:

$$A = \{1,2,4\}, \qquad B = \{2,4,6\}, \qquad C = \{3,5\}, \qquad D = \{6\}$$

Then, events A, C and D form a partition of Ω, since

$$A \cap D = \emptyset \qquad A \cap C = \emptyset \qquad C \cap D = \emptyset$$

$$A \cup C \cup D = \{1,2,3,4,5,6\}$$

The Algebra of Events

Cardinality of a set is a *measure* of the "number of elements of the set".

$$\text{card}\, A \equiv |A|$$

For a finite set A, $\text{card}\, A =$ "number of elements in A “.

For infinite sets, their cardinality is expressed with ***transfinite numbers***.

More on transfinite numbers

Definition of Probability (*finite sample space*)

Consider a sample space $\Omega = \{\omega_1, \omega_2, \dots \omega_n\}$ and an event $A = \{a_1, a_2, \dots a_m\}, m \leq n$, $A \subseteq \Omega$.

Then, the probability of event A

$$\Pr(A) = P(A) = \frac{\text{card } A}{\text{card } \Omega} = \frac{m}{n}.$$

$$\frac{\text{number of outcomes favorable to A}}{\text{total number of outcomes}}$$

The Algebra of Events

Consider the experiment that consists in rolling a die twice and recording the number of points.

What is the probability that the total sum of points will be greater or equal to 10? exactly 10?

Sample space:

$$\Omega = \{(1;1),(1;2),\dots,(6;6)\}, \quad \text{card}\,\Omega = 36$$

Events:

$$A = \{(4;6),(5;5),(5;6),(6;4),(6;5),(6;6)\}$$

$$B = \{(4;6),(5;5),(6;4)\}$$

Then

$$\Pr(A) = \frac{6}{36} = \frac{1}{6}, \qquad \Pr(B) = \frac{3}{36} = \frac{1}{12}.$$

What about infinite sample spaces?

Instead of counting the number of elements, we assign a measure to the set – nonnegative value, intuitively interpreted as its size (length, area, volume, etc.).

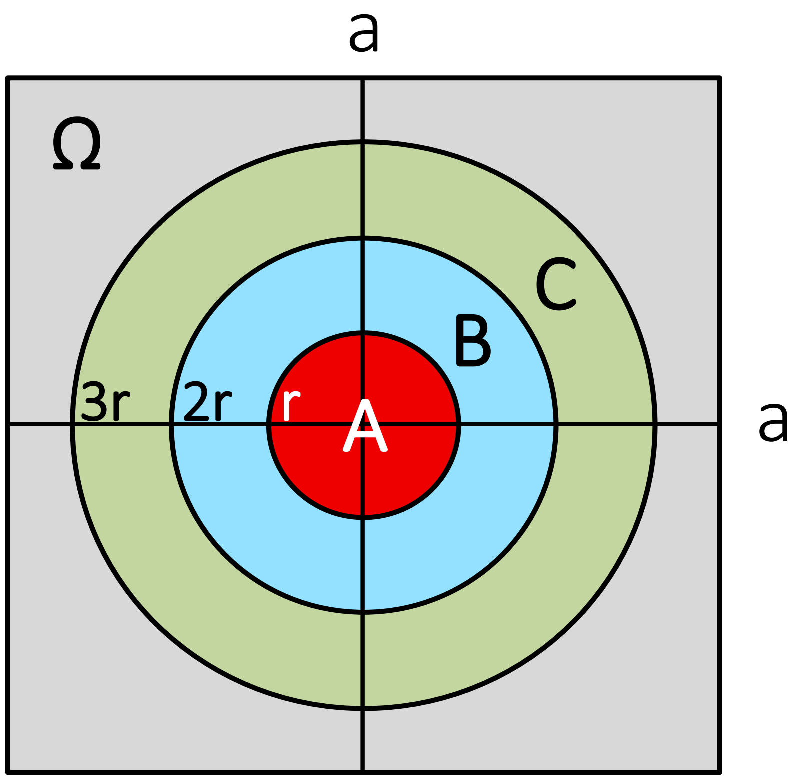
Then

$$\Pr(A) = \frac{\mathrm{mes}\, A}{\mathrm{mes}\, \Omega}$$

The Algebra of Events

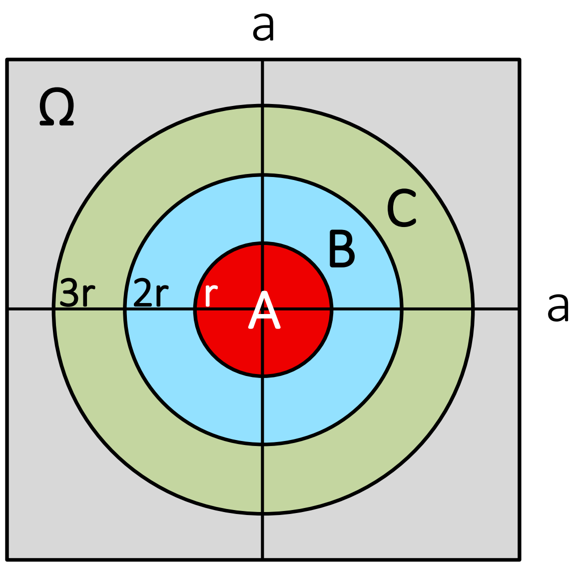
A shooter fires randomly at $a \times a$ square target which has three concentric circles of radii $r, 2r$ and $3r$.

What are the probabilities of hitting each colored zone (red, blue, green, and gray)?



What is the probability of hitting red or blue zone?

First, we must make an assumption that <u>all shots hit the $a \times a$ target</u> (no shots placed outside the square).



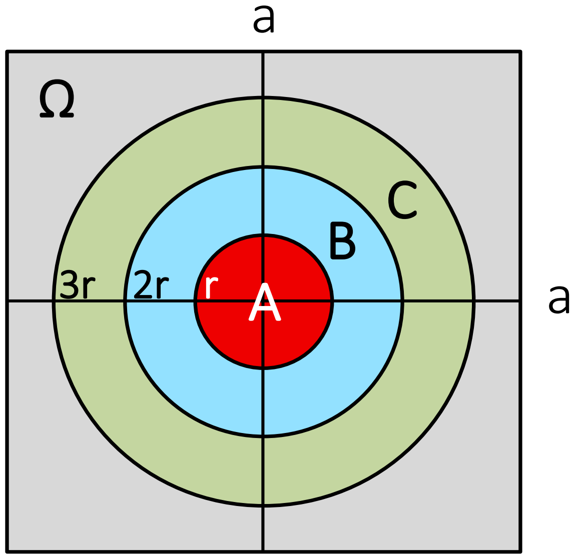
Thus, the points inside the target form the sample space Ω.

Since the target consists of infinite number of points, we should determine its measure (area):

$$\text{mes}\,\Omega = a^2$$

The Algebra of Events

Next, we should compute measures for each colored zone (subset, event).



$$\text{mes } A = \pi r^2$$

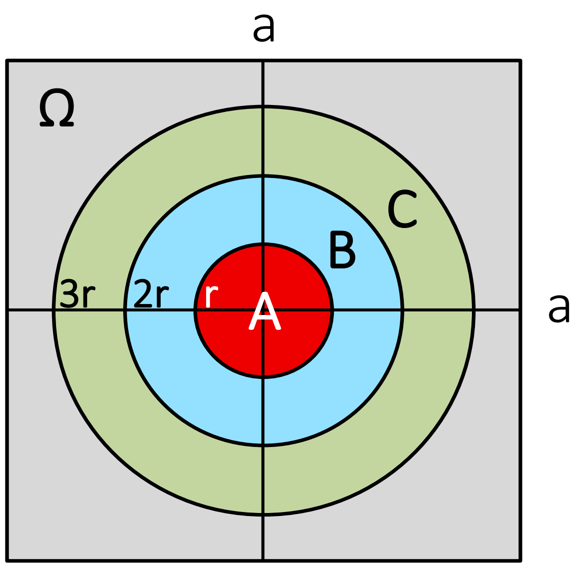
$$\text{mes } B = 4\pi r^2 - \pi r^2 = 3\pi r^2$$

$$\text{mes } C = 9\pi r^2 - 4\pi r^2 = 5\pi r^2$$

$$\text{mes } (\Omega \setminus A \cup B \cup C) = a^2 - 9\pi r^2$$

$$\text{mes } (A \cup B) = \pi r^2 + 3\pi r^2 = 4\pi r^2$$

The Algebra of Events



Finally, we can obtain the probabilities of hitting each colored zone.

$$\Pr(A) = \frac{\text{mes } A}{\text{mes } \Omega} = \pi \cdot \frac{r^2}{a^2}$$

$$\Pr(B) = \frac{\text{mes } B}{\text{mes } \Omega} = 3\pi \cdot \frac{r^2}{a^2}$$

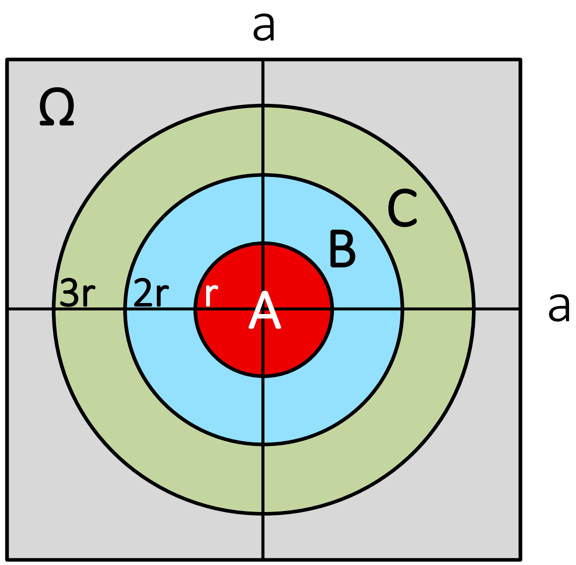
$$\Pr(C) = \frac{\text{mes } C}{\text{mes } \Omega} = 5\pi \cdot \frac{r^2}{a^2}$$

$$\Pr(\Omega \setminus A \cup B \cup C) = \frac{\text{mes } (\Omega \setminus A \cup B \cup C)}{\text{mes } \Omega} = 1 - 9\pi \cdot \frac{r^2}{a^2}$$

The Algebra of Events

… and the probability of hitting red or blue zone:

$$\Pr(A \cup B) = \frac{\text{mes}\,(A \cup B)}{\text{mes}\,\Omega} = 4\pi \cdot \frac{r^2}{a^2}$$



Note, that events A, B, C and $\Omega \setminus A \cup B \cup C$ are mutually exclusive since we can’t hit two different colors simultaneously.

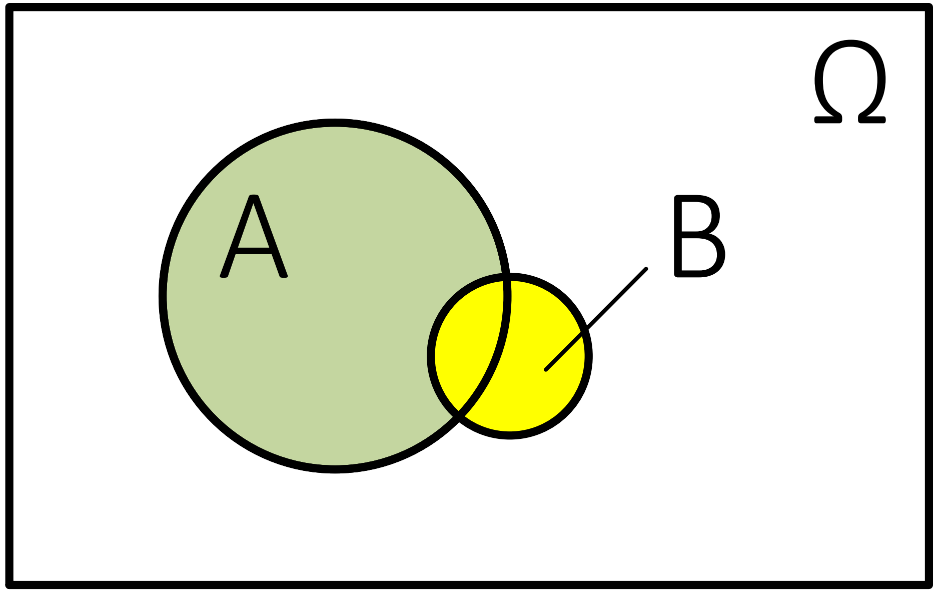
The Algebra of Events

Axioms of Probability

- $0 \leq \Pr(A) \leq 1$
- $\Pr(\Omega) = 1$
- if $A = A_1 \cup A_2 \cup \cdots \cup A_n$, where $A_1, \ldots, A_n$ are mutually exclusive events, then

$$\Pr(A) = \sum_{i=1}^{n} \Pr(A_i)$$

Probability of Union



$B \nsubseteq A$

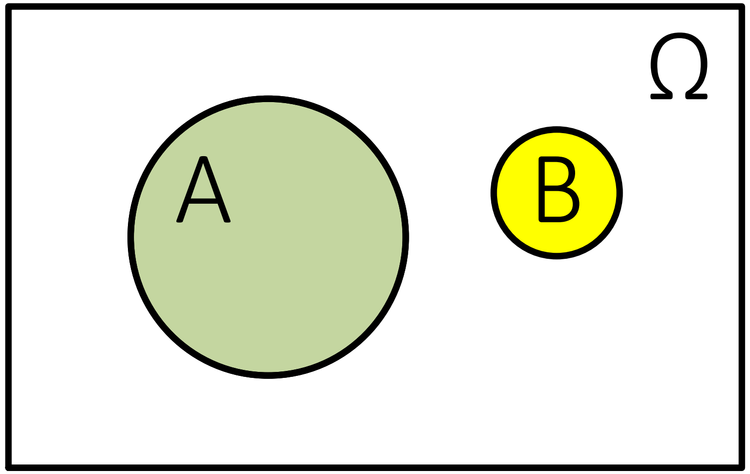
$A \cap B \neq \emptyset$

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

The Algebra of Events

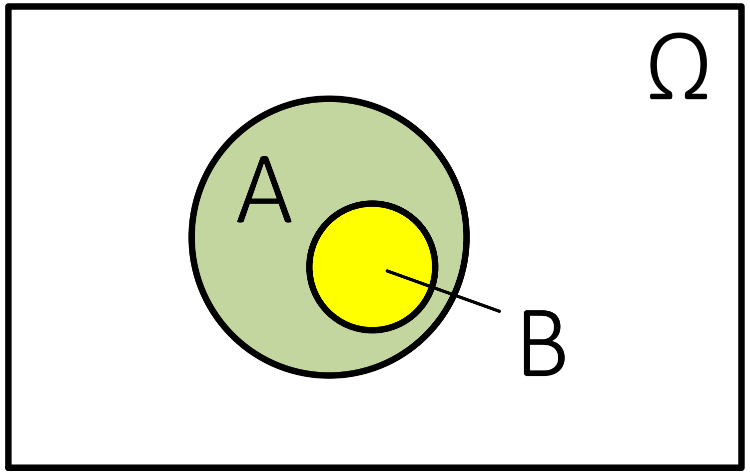
Probability of Union

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$



$$A \cap B = \emptyset$$

$$\Pr(A \cup B) = \Pr(A) + \Pr(B)$$



$$B \subseteq A \qquad A \cap B = B$$

$$\Pr(A \cup B) = \Pr(A)$$

Textbook Assignment

Géza Schay. *Introduction to Probability with Statistical Applications*

- **Chapter 2. 5-23 pp.**

F.M. Dekking et al. *A Modern Introduction to Probability and Statistics*

- **Chapter 2. 13-24 pp.**