The Law of Large Numbers

A factory manufactures three types of products:

- 30% with the price of 10$;
- 30 % with the price of 20$;
- 40% with the price of 30$.

- How many items should be randomly selected in order to be at least 90% sure their average price is in the $\pm 5\%$ range of the mean price?
- What is the probability that the total cost of these items is at most 3500$?

We start with determining the mean (expected) price of the items produced and its variance:

$$E[X] = 0.3 \times 10 + 0.3 \times 20 + 0.4 \times 30 = 21$$

$$Var(X) = E[X^2] - E[X]^2 = (0.3 \times 100 + 0.3 \times 400 + 0.4 \times 900) - 21^2 = 69$$

The Law of Large Numbers

According to the law of large numbers, if $\bar{X}_n$ is the average of n independent random variables with the same expectation μ and variance σ^2, then:

$$E[\bar{X}_n] = \mu, \qquad Var(\bar{X}_n) = \frac{\sigma^2}{n}.$$

Let's define the boundaries of the ±5% range :

Left – $0.95 \times 21 = 19.95$

Right - $1.05 \times 21 = 22.05$

So, we need to find such number n of items that

$$\Pr\{19.95 \le \bar{X}_n \le 22.05\} \ge 0.9$$

The Law of Large Numbers

We know the variance of the average price of n items is n times less than the variance of the single item, so:

$$Var(\bar{X}_n) = \frac{69}{n}$$

$$\sigma_{\bar{X}_n} = \sqrt{Var(\bar{X}_n)} = \frac{\sqrt{69}}{\sqrt{n}} \cong \frac{8.307}{\sqrt{n}}$$

According to the CLT, the new random variable

$$Z_n = \frac{\bar{X}_n - \mu}{\sigma_{\bar{X}_n}}$$

has the standard normal distribution, $N(0,1)$.

We also must transform the boundaries for our range:

Left - $\frac{19.95-21}{8.307} \times \sqrt{n} = \frac{-1.05}{8.307} \times \sqrt{n} = -0.126\sqrt{n};$

Right - $\frac{22.05-21}{8.307} \times \sqrt{n} = \frac{1.05}{8.307} \times \sqrt{n} = 0.126\sqrt{n}.$

In order to solve the first half of the problem, we need to find such n that

$$\Pr\{-0.126\sqrt{n} \leq Z_n \leq 0.126\sqrt{n}\} \geq 0.9$$

We know that $\Phi(-x) = 1 - \Phi(x)$, where Φ is the cdf of the standard normal distribution.

The probability that a random variable X assumes the value within the interval (α, β) is given by

$$\Pr\{\alpha \leq X \leq \beta\} = F_X(\beta) - F_X(\alpha).$$

Given that,

$$\Pr\{-0.126\sqrt{n} \leq Z_n \leq 0.126\sqrt{n}\} =$$

$$= \Phi(0.126\sqrt{n}) - \Phi(-0.126\sqrt{n}) = 2 \cdot \Phi(0.126\sqrt{n}) - 1$$

$$2 \cdot \Phi(0.126\sqrt{n}) - 1 \geq 0.9$$

$$2 \cdot \Phi(0.126\sqrt{n}) \geq 1.9$$

$$\Phi(0.126\sqrt{n}) \geq 0.95$$

To find the argument, we can consult the table of standard normal cdf (see Lecture 5), or compute with Mathcad:

$$qnorm(0.95,0,1) = 1.645$$

$$\Phi(0.126\sqrt{n}) \geq 0.95 \rightarrow 0.126\sqrt{n} \geq 1.645$$

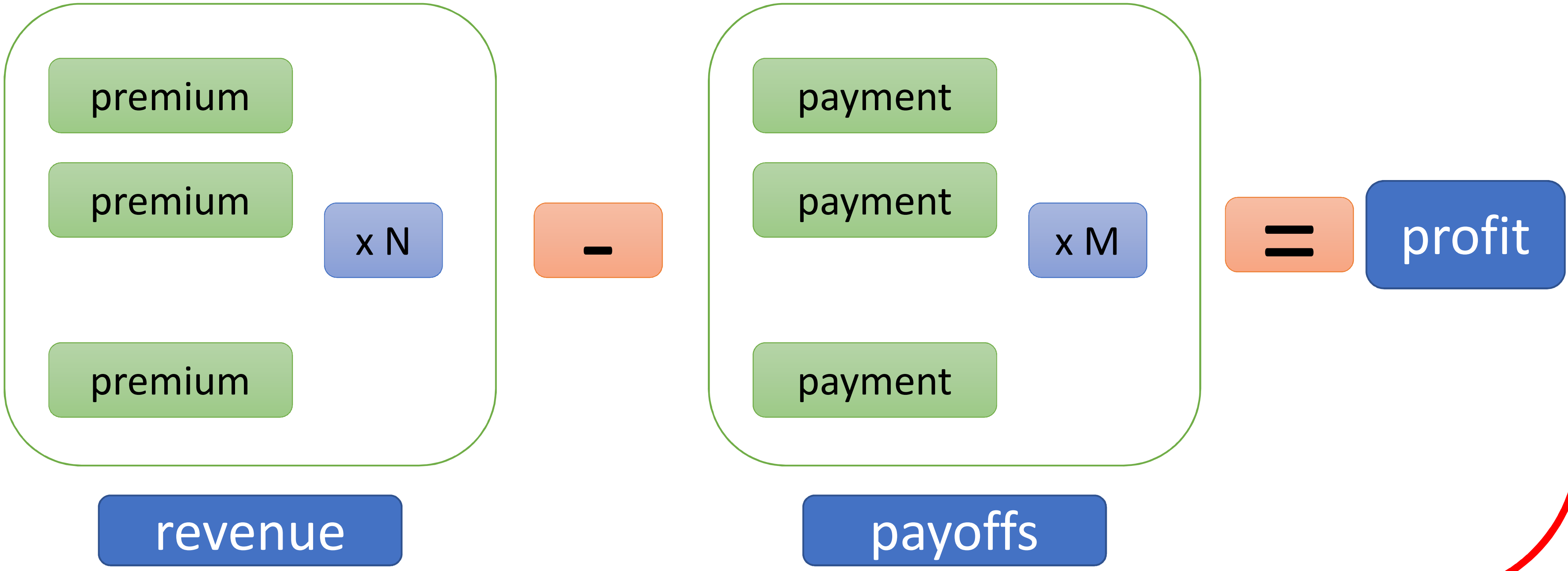
$$0.126\sqrt{n} \geq 1.645 \rightarrow n \geq 170.45$$

So, we must randomly select at least 171 items in order to ensure their average price is within the given range with the probability of greater or equal to 0.9.

Solve the second half of the problem by yourself.

The Law of Large Numbers

Oversimplified model of an insurance company



The Law of Large Numbers

The insurance company sold 300,000 policies; the premium was 2,000 rubles. The insurance claim payments were set at 1,500,000 rubles. The risk of insurable event is defined as 0.0013.

- What is the probability the company will lose money by the end of a year?
- What is the probability the company will have profit over 40,000,000 rubles?

De Moivre-Laplace limit theorem (see Lecture 4):

Consider a binomial experiment (Bernoulli trials) consisting of n trials with p – the probability of success, and $q = 1 - p$ – the probability of failure. The probability $P_n(m_1 \leq k \leq m_2)$ that number of successes would be $k \in [m_1; m_2]$ can be obtained by

$$P_n(m_1 \leq k \leq m_2) \approx \Phi(\beta) - \Phi(\alpha)$$

where

$$\alpha = \frac{m_1 - np}{\sqrt{npq}}, \qquad \beta = \frac{m_2 - np}{\sqrt{npq}}.$$

The revenue is the sum of all the premiums:

$$revenue = 300{,}000 \times 2{,}000 = 600{,}000{,}000\ rubles$$

The average (expected) number of insurable events is:

$$\mu = n \cdot p = 300{,}000 \times 0.0013 = 390$$

For the company to have no profit, the overall payoffs must be equal to the total revenue, that is,

$$\frac{revenue}{payment} = \frac{600{,}000{,}000}{1{,}500{,}000} = 400$$

Given that, the probability that the company will lose money is equal to the probability that 400 or more insurable events will occur:

$$Pr\{400 \leq X \leq 300{,}000\}.$$

According to de Moivre-Laplace limit theorem:

$$P_n(400 \leq k \leq 300{,}000) \approx \Phi(\beta) - \Phi(\alpha)$$

$$\alpha = \frac{400 - 390}{\sqrt{300{,}000 \times 0.0013 \times 0.9987}} = 0.507$$

$$\beta = \frac{300{,}000 - 390}{\sqrt{300{,}000 \times 0.0013 \times 0.9987}} = 15181.213$$

$$P_n(400 \leq k \leq 300{,}000) \approx \Phi(15181.213) - \Phi(0.507) =$$
$$= 1 - 0.694 = 0.306$$

Solve the second part of the problem by yourself.