Lection 6

Green function (sources method)

The physical sense of source method consists in following. The heat propagation process in a body can be presented as set of the temperature smoothing processes from many elementary sources distributed in space and in time. The problem solution using this method leads basically to the correct choosing of sources and their distribution.

Elementary source action in infinite body for one dimensional heat flux is characterized by formula

$$G(x,\xi,t)=\frac{\theta}{\sqrt{4\pi\kappa t}}\exp\left(-\frac{(x-\xi)^2}{4\kappa t}\right), \tag{1}$$

called Green function of source (Green Function) in infinite straight line. It is not difficult to test that the green function $G(x,\xi,t)$ satisfies to thermal conduction equation

$$\frac{\partial T}{\partial t}=\kappa\frac{\partial^2 T}{\partial x^2}. \tag{2}$$

Usually, this function $G(x,\xi,t)$ is called fundamental solution of thermal conduction equation and presents the temperature in the point x, if the heat quality $Q=\theta c\rho$ releases in initial time moment in the point ξ. The heat quantity in the straight line follows from

$$G(x,\xi,t)=\frac{c\rho\theta}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\exp\left(-\frac{(x-\xi)^2}{4\kappa t}\right)\frac{dx}{2\sqrt{\kappa t}}=\frac{c\rho\theta}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\exp\left(-u^2\right)du=c\rho\theta, \tag{3}$$

where $u=\dfrac{x-\xi}{2\sqrt{\kappa t}}$; $\displaystyle\int_{-\infty}^{+\infty}e^{-u^2}du=\sqrt{\pi}$.

For body of infinite sizes and one dimensional heat flux, we have the green function

$$G_l(x,\xi,t)=\frac{2\theta}{l}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi\xi}{l}\right)\exp\left(-\frac{n^2\pi^2}{l^2}\kappa t\right), \tag{4}$$

that shows the temperature distribution in infinite plate $(0<x<l)$ in time t, if temperature was equal zero in initial time moment and the heat quantity $Q=\theta c\rho$ releases simultaneously in the point ξ.

The simple example connects with temperature search in infinite body for arbitrary tome at the condition for initial time

$$T(x,0)=f(x). \tag{5}$$

The thermal conduction equation (2) is correct for any time; the sources absent in infinity

$$\frac{\partial T(+\infty,t)}{\partial x}=\frac{\partial T(-\infty,t)}{\partial x}=0.$$

The particular solution of (2) has a form (correspondingly to above) (

$$T=\frac{C}{\sqrt{4\pi\kappa t}}\exp\left(-\frac{(x-\xi)^2}{4\kappa t}\right). \tag{6}$$

The temperature has a maximum in the point $x=\xi$. We shift the coordinate origin to this point. The area under curve is the finite value and equals to integral on (6) in the limits from $-\infty$ to $+\infty$.

$$S=\int_{-\infty}^{+\infty}\frac{C}{\sqrt{4\pi\kappa t}}\exp\left(-\frac{(x-\xi)^2}{4\kappa t}\right)d(x-\xi)=\frac{C}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-z^2}dz=C.$$

Using the function properties, we change the curve $f(x)$ by infinite many curves of view

$$\lim_{t\to 0}\left[\frac{C}{\sqrt{4\pi\kappa t}}\exp\left(-\frac{(x-\xi)^2}{4\kappa t}\right)\right].$$

Then full initial distribution will equal

$$\lim_{t\to 0} T(x,t)=\lim_{t\to 0}\left[\frac{1}{\sqrt{4\pi\kappa t}}\int_{-\infty}^{+\infty} f(\xi)\exp\left(-\frac{(x-\xi)^2}{4\kappa t}\right)d\xi\right].$$

This relation is correct for any time. Hence the solution takes the form

$$T(x,t)=\frac{1}{\sqrt{4\pi\kappa t}}\int_{-\infty}^{+\infty} f(\xi)\exp\left(-\frac{(x-\xi)^2}{4\kappa t}\right)d\xi. \qquad (7)$$

It is not difficult to generalize this method for two and three dimensional problems. This method is applied to practical problems, when the surface treatment is carried out by moving sources (for example, when the cutting and welding processes ate described).

Method of separation of variables

We illustrate the method of separation of variables with the help of examples.

It is necessary to find the solution of the problem

$$\frac{\partial T}{\partial t}=\alpha^2\frac{\partial^2 T}{\partial x^2} \qquad (1)$$

$$0<x<1;\ 0<t<\infty$$

$$T(0,t)=0;\ T(1,t)=0 \qquad (2)$$

$$T(x,0)=\varphi(x);\ 0\le x\le 1 \qquad (3)$$

(thermal diffusivity coefficient κ is signed as α^2). It is the problem on the plate of unit thickness cooling.

We look for the solution in the form

$$T=X(x)\theta(t) \qquad (4)$$

Substituting (4) in (1), we come to the equations

$$\frac{\theta'(t)}{\alpha^2\theta(t)}=\frac{X''(x)}{X(x)}=k \qquad (5)$$

So the variables x and t are independent, we have instead (1) two equations

$$\theta'-k\alpha^2\theta=0; \qquad (6)$$

$$X''-kX=0. \qquad (7)$$

Boundary conditions to (7) are written as

$$X(0)=0;\ X(1)=0, \qquad (8)$$

that follows from (2).

The problem (7), (8) is eigenvalue problem (Sturm-Liouville problem).

Mark that $k<0$; in opposite case the problem (7), (8) will have only trivial solution. The function $\theta(t)$ must diminishes for $t\to\infty$, that is $k=-\lambda^2\neq 0$.

The solutions of the equations (6), (7) can be found by usual method:

$$\theta(t)=Ce^{-\lambda^2\alpha^2 t},$$

$$X(x)=A\sin(\lambda x)+B\cos(\lambda x),$$

where A,B,C are arbitrary constants.

Hence

$$T(x,t)=\exp\left(-\lambda^2\alpha^2 t\right)\left[A\sin(\lambda x)+B\cos(\lambda x)\right].$$

Substituting that in the condition (2), we shall find:

$$T(0,t) = B\exp\left(-\lambda^2\alpha^2 t\right) = 0 \text{, if } B = 0 .$$

$$T(1,t) = A\exp\left(-\lambda^2\alpha^2 t\right)\sin(\lambda) = 0 \text{, if } \sin\lambda = 0$$

The last gives the restrictions for values of λ:

$$\lambda = \pm\pi, \pm 2\pi, \ldots, \text{ or } \lambda_n = \pm n\pi, n = 1,2,\ldots$$

Then $T_n(x,t) = A_n \exp\left(-(\pi n\alpha)^2 t\right)\sin(\pi nx) = 0$, $n = 1,2,\ldots$ and

$$T(x,t) = \sum_{n=1}^{\infty} A_n \exp\left(-(\pi n\alpha)^2 t\right)\sin(\pi nx) \tag{9}$$

From initial conditions, we have

$$\varphi(x) = \sum_{n=1}^{\infty} A_n \sin(\pi nx) \tag{10}$$

The function system $\{\sin(\pi nx), n = 1,2,\ldots\}$ obeys the property of orthogonality, that is

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)dx = \begin{cases} 0, & m \neq n; \\ 1/2, & m = n. \end{cases} \tag{11}$$

Using this property, we come to the solution

$$T(x,t) = \sum_{m=1}^{\infty} A_m \exp\left(-(\pi m\alpha)^2 t\right)\sin(\pi mx), \tag{12}$$

where

$$A_m = 2\int_0^1 \varphi(\xi)\sin(\pi m\xi)d\xi . \tag{13}$$

It is no difficult to obtain from (12), (13) the solution in the form

$$T(x,t) = \int_0^1 G(x,\xi,t)\varphi(\xi)d\xi ,$$

where

$$G(x,\xi,t) = 2\sum_{m=1}^{\infty} \exp\left(-(\pi m\alpha)^2 t\right)\sin(\pi mx)\sin(\pi m\xi) \text{ -}$$

Green function for instantaneous source of capacity $Q = c\rho$.

This method with some modifications can be used for more complex problem solution, for example:

$$\begin{gathered} \frac{\partial T}{\partial t} = \alpha^2 \frac{\partial^2 T}{\partial x^2} + f(x,t) \\ \alpha_1 \frac{\partial T(0,t)}{\partial x} + \beta_1 T(0,t) = g_1(t) \\ \alpha_2 \frac{\partial T(L,t)}{\partial x} + \beta_2 T(L,t) = g_2(t), \\ T(x,0) = \varphi(x). \end{gathered} \tag{14}$$

Radiation heat transfer

Heat radiation is the process of internal energy propagation of emitting body by electromagnetic oscillations and photons. Any bodies with temperature more the absolute zero emit the electromagnetic oscillations. Electrons, ions entering into the composition of substance are generators of electromagnetic waves. Additionally to wave properties, the radiation obeys corpuscular properties. That is the energy emits and absorbs by substances by discrete portions – photons.

The heat radiation intensity depends on material and body temperature, surface state, and for gases, on the thickness of the layer and pressure. The radiation energy increases with temperature growth because the body internal energy increases. At high temperatures, heat radiation can be basic mechanisms of heat transfer because radiation intensity depends on the temperature more strongly then convection and thermal conduction ones.

As opposed to another ways of heat exchange, radiation energy flux is transparent both on more heated body to less heated body and on the contrary ones. Finally result of this interrelation consists in the heat quantity transferred by radiation.

All types of radiation differ by wave length.

Summary energy emitted from the body surface in full interval of wave lengths during unit time is called integral or full radiation flux. This value is measured in Watt – W:

$$Q = \int_F E dF , \qquad (15)$$

where E is energy emitted from unit body surface during unit time in all directions of half – spherical space, W/cm^2.

The value E depends only on temperature and physical properties of the body and is called self-radiation or body transmissibility. This value is flux density of integral radiation

$$E = q .$$

Consider the body participates in the radiation heat exchange with other bodies (Fig. 1). The radiation energy Q falls on the surface of given body from the other body's. This energy is absorbed by body partly, partly reflects and partly passes through the body. Each from these parts is characterized by corresponding flux – the flux of absorbed radiation Q_A; the flux of reflected radiation Q_R; the flux of emitted radiation Q_D. We can write down

$$Q_A = AQ ; \; Q_R = RQ ; \; Q_D = DQ ,$$

where A is absorptance of body; R is radiant reflectance of body; D is transmittance of the body.

Corresponding to energy conservation law, integral radiant flux falling on the body equals to a sum all parts

$$Q = Q_A + Q_R + Q_D . \qquad (16)$$

It if follows from (16)

$$A + R + D = 1 . \qquad (17)$$

Each from these coefficients can change in the limits from 0 to 1.

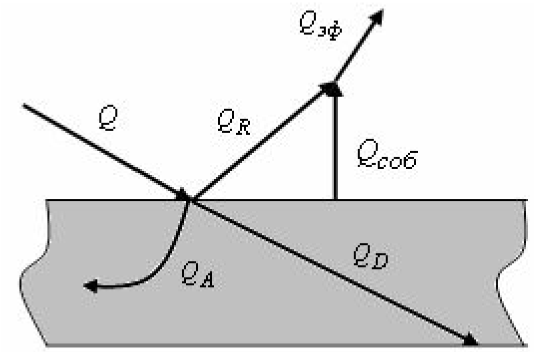

Fig.1. Constituents of integral radiation flux.

If absorptance of body $A = 1$, two other coefficients equal to zero ($R = D = 0$). The bodies, absorbed all falling energy, are called ***absolutely black bodies***.

The body with $R = 1$ and, correspondingly $A = D = 0$, reflects all radiation energy. If the reflection occurs by laws of geometrical optics, then its surface is called ***mirror surface***; if the reflection is scattered radiation, then its surface is called ***absolutely white***.

The body with $D = 1$, and $A = R = 0$, transmits all radiant energy. It is called the absolutely transparent body. The body's with $0 < D < 1$, are called half – transparent ones.

The sum of self radiation and reflected radiation gives the ***flux of effective body radiation*** (Fig. 1)

$$Q_{eff} = Q_{self} + Q_R . \qquad (18)$$

So, the combined process of emission, absorption, reflection and transmission of the radiant energy in the system of various bodies is radiation heat exchange The bodies can have the same or different temperatures.

There are not absolutely white and black bodies in the reality. these notions are conditional.

The basic laws of heat radiation

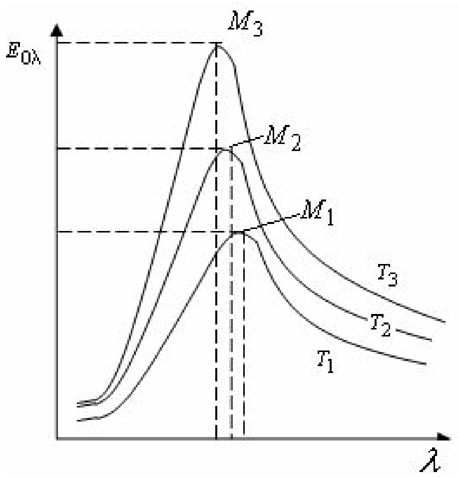

Fig. 2

Planck law. In year 1900 M.Plank, on the basis of electromagnetic nature of radiation and developed them quantum theory, establishes for absolutely black body (index 0) the dependence of self-radiation intensity on the wave length and temperature

$$E_{0\lambda} = \frac{C_1 \lambda^{-5}}{e^{C_2/(\lambda T)} - 1}, \qquad (19)$$

where λ is wave length, m; T is absolute body temperature, K; C_1, C_2 are constant: $C_1 = 3{,}74 \cdot 10^{-16}$ W·m²; $C_2 = 0{,}0144$ m·K. This law is illustrated with the help of Fig.2.

Wien displacement law. From Planck law one can obtain the wave length corresponding to maximal density of radiant flux solving the equation

$$dE_{0\lambda} / d\lambda = 0 . \qquad (20)$$

We obtain the formulae

$$\lambda_{max} = 2{,}9 \cdot 10^{-3} / T , \qquad (21)$$

presenting the mathematical formulation of Wien law.

It is evidently that maximal spectral density of radiant flux shifts to the side of mire short lengths of wave.

One can see from Fig.2 that if $T_3 > T_2 > T_1$, so $\lambda_{3,max} > \lambda_{2,max} > \lambda_{1,max}$.

Stefan-Boltzmann law, open in year 1879 by Czech scientist I. Stefan and based theoretically in year 1884 by Austrian scientist L. Boltzmann establishes the dependence of emissivity of absolutely black body on its temperature:

$$E_0 = \int_0^\infty E_{0\lambda} d\lambda = \sigma_0 T^4 , \qquad (22)$$

where $\sigma_0 = 5{,}77 \cdot 10^{-8}$ W/(m²K⁴) is Stefan-Boltzmann constant.

So, the density of radiant flux of absolutely black body is proportional to four degree of its absolute temperature.

Integral emission of absolutely black body at given temperature in the limits from $\lambda = 0$ to $\lambda = \infty$ is presented graphically by the area restricted the curve for $T = const$ and abscissa-axis.

Sometimes, ones present the law (22) in the form

$$E_0 = C_0 (T/100)^4, \quad (23)$$

where $C_0 = 5{,}77$ W/(m^{2}K^4) is the emissivity of absolutely black body.

For real bodies, that is for no absolutely black bodies (gray bodies), the flux density of radiation is expressed by the same formulae

$$E = C(T/100)^4,$$

but the value C relates to the gray bodies.

To compare the flux densities of real and absolutely black body for the same temperature the body characteristics ε called by blackness degree is used

$$\varepsilon = E/E_0 = C/C_0$$

The Stefan-Boltzmann law for gray body is described as

$$E = \varepsilon\sigma_0 T^4 = \varepsilon C_0 (T/100)^4.$$

The value ε for gray bodies as less then unity; it depends on body nature, surface state, temperature and is found from experiment.

Lambert's law. Radiant energy distribution emitted by absolutely black body in various directions is no identical. In year 1760 German scientist I Lambert establishes the dependence of the value of radiant energy on propagation direction. Mathematical formulation of Lambert's law for density of radiant energy in the direction m made an angle φ with the normal n to the emitting area has the form (Fig. 3)

$$E_{0\varphi} = E_{0n} \cos\varphi, \quad (24)$$

where E_{0n} is the radiant flux density of absolutely black body in the direction of normal to the surface $(\varphi = 0)$.

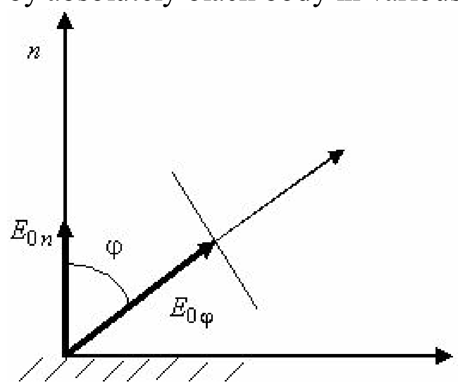

Fig. 3.

Kirchhoff law establishes 3the interrelation between the body abilities to emission and absorption of energy. This interrelation can be obtained from thermodynamical equilibrium at radiant heat exchange between two parallel surface.

Kirchhoff law is formulated in following manner: the relation of radiant flux density of gray body to their absorbability does not depend on body nature and equals to the radiant flux density of absolutely black body at the same temperature.

At the thermodynamical equilibrium absorbability and blackness degree are equal to each other

$$A = \varepsilon$$

In real conditions, the radiant heat transfer is accompanied by other forms of heat transfer – by convection and thermal conduction. Such combined heat transfer process is called complex heat exchange. It is very important to evaluate the contribution of each part of heat exchange in the real conditions.

The ***Lambert-Bouguer law*** describes the gradual weakening of parallel monochromatic of optical beam in absorbing substance:

$$q_A = q_0 \exp(-\kappa x) = AE \exp(-\kappa x), \quad (25)$$

where κ is the absorption index depending on substance nature and state and on wave length of transmitted light.

Lambert-Bouguer-Beer law connects the light weakening with the absorbing centers presence and mathematically follows from the assumption that the relative light weakening for infinite thin layer does not depend on light intensity and proportional to the concentration of absorbing substance and thickness of this layer dx

$$\frac{dq}{q} = -\kappa_0 C dx \,.$$

The assumptions are approximate.