Lection 5

Introduction

To solve complex problems it is necessary to know rigorous mathematical methods. We stop only on exact methods. Green function method, Dyugamel-Neyman method, variable separation method, Integral transform methods belong to them. The study of the methods in details must be carried out as special school discipline. The some examples are useful for us. The one of classification the thermal conduction problems is presented on the Fig. 1

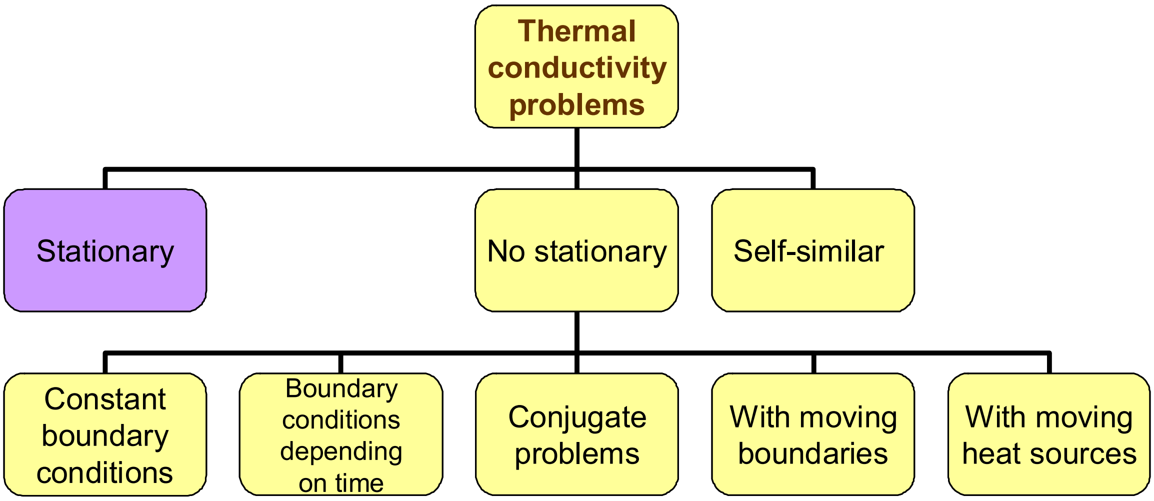

Fig. 1.

The simplest non stationary problems

Until now we studied the systems where the thermal conduction processes was completed. But some time should pass to stationary state establishment. Many technology processes of material treatment and production are non stationary. We stop on the analysis of some no stationary problems without study theoretical basis of exact analytical methods. The simplest non stationary problems can call zero-dimension.

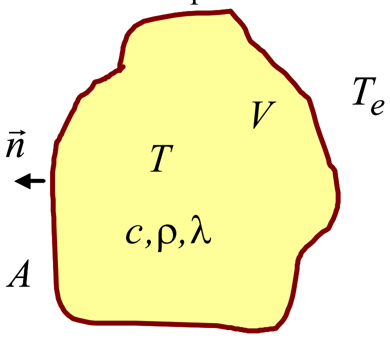

Fig. 2. Illustration to the simplest problem

Let there is the body with very high thermal conductivity coefficient. In this case the small temperatures gradients allow neglect the temperature distribution in the body. Full mathematical problem formulation includes three dimensional thermal conduction equation (Fig. 2)

$$c\rho \frac{\partial T}{\partial t} = \lambda \Delta T \tag{1}$$

and boundary and initial conditions

$$-\left(\frac{\partial T}{\partial n}\right)_A = \alpha (T - T_e), \tag{2}$$

$$t = 0 : T = T_0, \tag{3}$$

Using the described conditions we come from (1) to heat balance equation

$$c\rho V \frac{\partial T}{\partial t} = -\alpha A (T - T_e), \tag{4}$$

where V is body volume, and A is its surface area. Environment temperature T_e is constant.

The correctness condition of this approximation can be written as

$$\frac{L}{\sqrt{\lambda t_o/(c\rho)}} << 1,$$

where t_o is some specific observation time.

In dimensionless variables

$$\theta = \frac{T - T_e}{T_0 - T_e}, \quad \tau = \frac{t}{t_*}, \tag{5}$$

where $t_* = \frac{L^2}{\kappa}$, $\kappa = \frac{\lambda}{c\rho}$ (that is $\tau = Fo$), L is specific linear size of the body, $L = V / A$, we come to the problem

$$\frac{d\theta}{d\tau} = -\frac{\alpha A t_*}{c\rho V}\theta = -\frac{\alpha L}{\lambda}\theta = -Bi\theta, \quad \tau = 0 : \theta = 1, \tag{6}$$

where $Bi = \frac{\alpha L}{\lambda}$ is Biot criterion known from previous lections

This problem is integrated very simple:

$$\frac{d\theta}{\theta} = -Bi d\tau \text{ or } \theta = \mathbf{exp}(-Bi \cdot Fo). \tag{7}$$

Employment of dimensionless criteria allows presenting the result for all bodies with any coefficients in integrated picture (Fig.3).

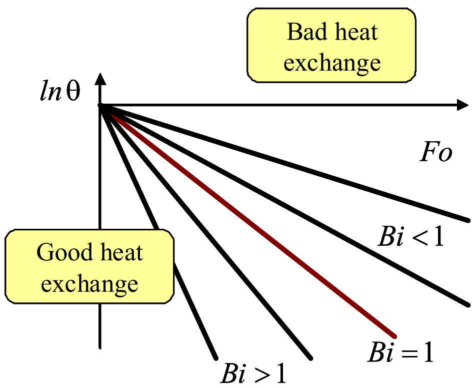

Fig. 3

Let now the **boundary conditions** is the **function of the time**. Let the temperature of the stream changes linearly

$$T_e = Bt + T_0.$$

Then the heat balance equation takes the form

$$c\rho V \frac{\partial T}{\partial t} + \alpha A T = \alpha A (Bt + T_0),$$

and the same initial conditions take a place.

In variables $\theta = \left(\frac{T - T_0}{T_* - T_0}\right)$ and τ, where T_* is not determined now, we have the problem

$$\frac{d\theta}{d\tau} + Bi\theta = Bi \cdot \beta \cdot \tau, \tag{8}$$

$$\tau = 0 : \theta = 0,$$

where $\beta = \frac{Bt_*}{T_* - T_0}$. Taking $\beta = 1$, we determine the scale temperature $T_* = T_0 + \frac{BL^2}{\kappa}$.

The solution of this problem takes the form

$$\theta = Fo - \frac{1}{Bi}\left[1 - \mathbf{exp}\left(-Bi \cdot Fo\right)\right]. \tag{9}$$

Therefore the detail (body) temperature falls behind medium temperature changing corresponding to low

$$\theta_e = \frac{T_e - T_0}{T_* - T_0} = Fo\,.$$

The simplest boundary problems

The *problem with boundary condition of first kind* takes a place, when the surface temperature is given. Let the body is half infinite (fig.1). This assumption is correct when the specimen size H is much grate then the heat zone forming in specimen during treatment

$$H >> \sqrt{\lambda t_* / (c\rho)}\,.$$

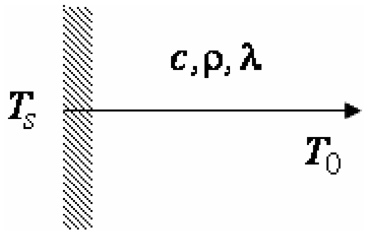

Fig.4. First boundary problem for half-space

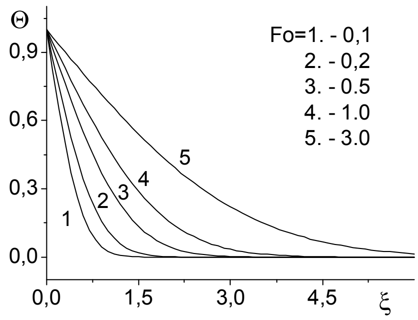

Fig. 5. Illustration the first boundary problem solution

Body thermal physical properties c, ρ, λ are constant. Mathematical formulation of the problem has a view

$$c\rho\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}; \tag{10}$$

$$x = 0:\ T = T_s;$$

$$x \to \infty:\ T = T_0;$$

$$t = 0:\ T = T_0.$$

Go to variables

$$\theta = \frac{T - T_0}{T_s - T_0},\ \tau = \frac{t}{t_*},\ \xi = \frac{x}{x_*},\ x_* = \sqrt{\kappa t_*}\,.$$

Then we shall find

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \xi^2};$$

$$\xi = 0:\ \theta = 1;$$

$$\xi \to \infty:\ \theta = 0;$$

$$\tau = 0:\ \theta = 0.$$

This problem in dimensionless variables does not contain the parameters. Its exact analytical solution takes the form

$$\theta = erfc\left(\frac{\xi}{2\sqrt{\tau}}\right), \tag{11}$$

where the function функция $erfc(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \mathbf{exp}\left(-t^2\right) dt$ is probability integral. This function has the properties

$$erfc(0) = 1;\ erfc(\infty) = 0;$$
$$z << 1:\ erfc(z) \approx \frac{2}{\sqrt{\pi}} z;$$
$$z >> 1:\ erfc(z) \approx \frac{\mathbf{exp}\left(-z^2\right)}{z\sqrt{\pi}}.$$

Qualitative temperature distribution for substances of any properties is shown on the Fig. 5.

It is very convenient for the solution of similar linear problems the ***Laplace integral transformation method.***

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The ***problem with the boundary condition of second kind*** appears when the heat flux acts on the body surface. If the energy density in flux is disturbed uniform along the surface of large specimen, we can assume that the temperature depends on one coordinate.

Let the flux of constant intensity q_0 falls on the surface of half-space (Fig. 6). Then the condition of constant temperature in the problem (10) is changed on the condition

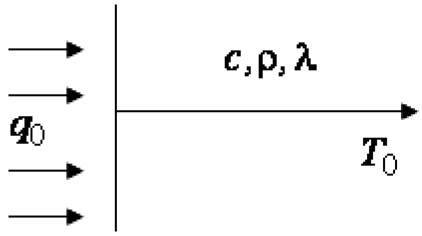

Fig. 6. Second boundary problem for half-space

$$-\lambda \frac{\partial T}{\partial x} = q_0 \tag{12}$$

In variables θ, τ, ξ, where temperature scale is not determined for a while we come to the problem

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \xi^2};$$
$$\xi = 0:\ -\frac{\partial \theta}{\partial \xi} = \frac{q_0\sqrt{\kappa t_*}}{\lambda\left(T_* - T_0\right)}; \tag{13}$$
$$\xi \to \infty:\ \theta = 0;$$
$$\tau = 0:\ \theta = 0.$$

Assuming

$$\frac{q_0\sqrt{\kappa t_*}}{\lambda\left(T_* - T_0\right)} \equiv \frac{q_0\sqrt{t_*}}{\sqrt{c\rho\lambda}\left(T_* - T_0\right)} \equiv 1,$$

we determine the temperature scale and obtain the problem without parameters.

This problem can be solved using method of Laplace integral transform.
We have the solution

$$\theta = 2\sqrt{\frac{\tau}{\pi}}\,\mathbf{exp}\left(-\frac{\xi^2}{4\tau}\right) - \xi erfc\left(\frac{\xi}{2\sqrt{\tau}}\right). \tag{14}$$

Surface temperature change as $\sqrt{\tau}$ ^

$$\xi = 0: \ -\frac{\partial\theta}{\partial\xi} = 1; \ \theta = 2\sqrt{\frac{\tau}{\pi}}.$$

The solution is illustrated on the figure. 7.

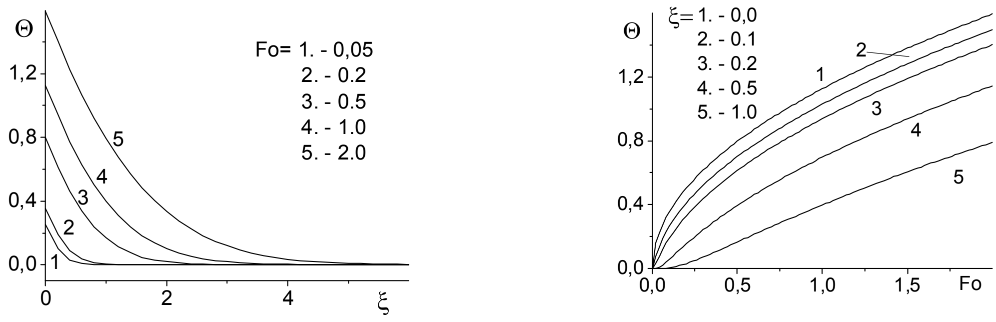

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Fig. 7. Illustration the second boundary problem solution: a) temperature distribution for different time moment; б) dependence of the temperature on time in different points.

Dugamel method

We illustrate Dugamel method with help of example. It is necessary to find the solution of the problem

$$c\rho\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)$$

$$\begin{aligned} &t = 0: \ T = T_0; \ x \geq 0 \\ t > 0: \quad &x = 0: \ T = f(t); \\ &x \to \infty: \ T = T_0 \end{aligned} \tag{15}$$

Assuming $T = u + \mathrm{v}$, we come to the problems for two functions, which must satisfy the same equation but more simple conditions:

$$\begin{aligned} &u = T_0, \ \ t = 0, x \geq 0; \\ &u = 0, \ \ \ x = 0, t > 0 \end{aligned} \tag{16}$$

and

$$\begin{aligned} &\mathrm{v} = 0, \ \ \ \ t = 0, x \geq 0; \\ &\mathrm{v} = f(t), \ \ x = 0, t > 0 \end{aligned} \tag{17}$$

The solution for function u follows from the solution of first boundary problem, described in physical variables. We have

$$u = T_0\left[1 - erfc\left(\frac{x}{2\sqrt{\kappa t}}\right)\right] = T_0 erf\left(\frac{x}{2\sqrt{\kappa t}}\right). \tag{18}$$

To find the solution of the problem for function v, we shall use the Dugamel theorem:

If $\varphi(x,t)$ is the solution for temperature change in solid body with zero initial temperature and surface temperature equal to unity, so the solution $\mathrm{v}(x,t)$ for the case of variable surface temperature, $\mathrm{v}(0,\mathrm{t}) = f(t)$, $x = 0$, follows from the formula

$$\mathrm{v}(x,t) = \int_0^t f(y) \frac{\partial}{\partial t} \varphi(x, t - y) dy . \qquad (19)$$

This theorem can be used for other problems also.

We write down the finally solution

$$T = T_0 + Ct\left[\left(1 + 2X^2\right) erfc(X) - \frac{2}{\sqrt{\pi}} X e^{-X^2}\right], \; X = \frac{x}{2\sqrt{\kappa t}} . \qquad (20)$$

Conjugate problems

The conjugate problems are widespread at the description of technology problems. For example, when the thermal treatment of the specimen is carried out in the medium with the given properties (Fig. 8), it is necessary to evaluate the role of the medium as compared with the vacuum. We come to the problem including the thermal conductivity equation equations for specimen and for the medium:

$$c_1 \rho_1 \frac{\partial T_1}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_1 \frac{\partial T_1}{\partial x}\right), \; x < 0$$

$$c_2 \rho_2 \frac{\partial T_2}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_2 \frac{\partial T_2}{\partial x}\right), \; x > 0 .$$

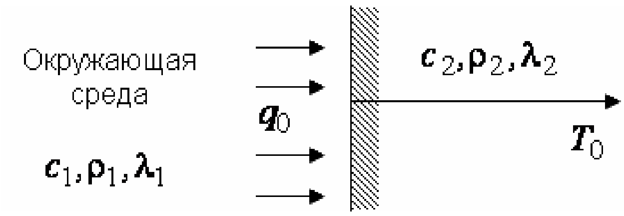

Fig. 8.

The boundary conditions of fourth kind are written don in interface

$$x = 0: \; \lambda_1 \frac{\partial T_1}{\partial x} - \lambda_2 \frac{\partial T_2}{\partial x} = q_0, \; T_1 = T_2 .$$

In infinity we have

$$x \to \pm\infty : \; T_i = T_0, i = 1{,}2$$

at the initial time moment -

$$t = 0 : \; T_i = T_0, i = 1{,}2 .$$

The problem on the thermal treatment of detail with the coating is the second example (Fig.9). The condition on the coating surface depends on the treatment condition.

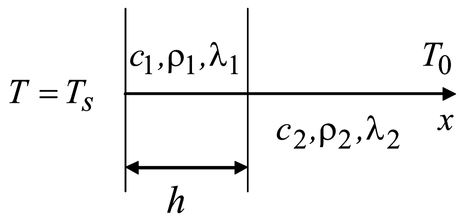

Fig.9

The solutions of conjugate problems depends on new parameters characterizing the thermal properties one material relatively to other one

$$K_c = \frac{c_1 \rho_1}{c_2 \rho_2}; \; K_\lambda = \frac{\lambda_1}{\lambda_2} .$$

These problems can be solved using Laplace integral transform method.