Lection 4

Thermal conduction equation in various coordinate systems

Thermal conductivity equation in the form

$$c\rho\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)+q_V \qquad (1)$$

is suitable for Cartesian coordinate system. In praxis, the conditions are found that lead to necessity to write thermal conduction equation in other form more convenient for problem solution and physical treatment. That relates for example to the bodies with the forms of rotation figures. The items expressing the heat release and energy accumulation are invariant relatively to coordinate system, but the items corresponding to resulting conductive flux depend on geometry. There is the presentation form for differential equations in which the equations are invariant relatively coordinate system change. We stop on cylindrical and spherical coordinate systems additionally to (1).

The dependence of the equation form on coordinate system vanish if the conductive items expresses through Laplace operator

$$\frac{1}{a}\frac{\partial T}{\partial t}=\Delta T+\frac{q_V}{\lambda}, \qquad (2)$$

where $a=\lambda/(c\rho)$. Instead Δ ones used often the symbol ∇^2. The record (2) is correct quite if thermal conductivity coefficient is constant. If this condition is not hold, the equation is presented in the form

$$c\rho\frac{\partial T}{\partial t}=div(\lambda gradT)+q_V, \qquad (3)$$

where $div\equiv\nabla\cdot$ is designation for divergence-operator (обозначение оператора дивергенции («nabla with point»); $grad\equiv\nabla$ is designation for gradient-operator.

Form of operators depends on coordinate system. For Cartesian coordinate system (Fig.1), for example, we have $T=T(x,y,z,t)$ and

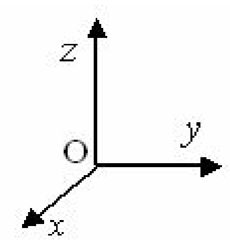

Fig. 1.

$$\Delta\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2};$$

$$\nabla\neq\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}$$

that is

$$\mathbf{q}=-\lambda\left(\frac{\partial T}{\partial x}\mathbf{i}+\frac{\partial T}{\partial y}\mathbf{j}+\frac{\partial T}{\partial x}\mathbf{k}\right)=-\lambda grad\,T$$

In cylindrical coordinate system (Fig. 2) we have $T=T(r,\varphi,z,t)$. Coordinates x and y connect with coordinates r and φ of cylindrical coordinate system by simple relations

$$x=r\cos\varphi;\ y=r\sin\varphi. \qquad (4)$$

Operators of Laplace and gradient have a view

$$\Delta=\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2};\ \nabla=\mathbf{i}_r\frac{\partial}{\partial r}+\mathbf{i}_\varphi\frac{1}{r}\frac{\partial}{\partial\varphi}+\mathbf{i}_z\frac{\partial}{\partial z}.$$

Therefore thermal conduction equation (3) takes the form

$$a\frac{\partial T}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial\varphi^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q_V}{\lambda}, \qquad (5)$$

and components of heat flux vector – the form

$$q_r = -\lambda \frac{\partial T}{\partial r};\ q_\varphi = -\lambda \frac{1}{r}\frac{\partial T}{\partial \varphi};\ q_z = -\lambda \frac{\partial T}{\partial z}. \qquad (6)$$

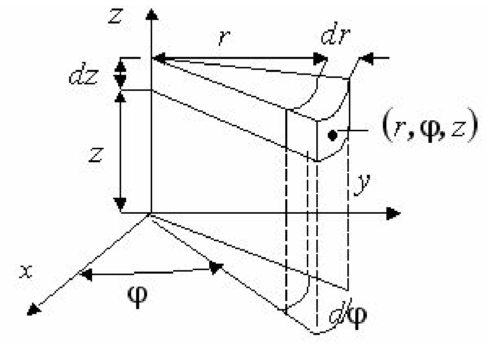

Fig. 2.

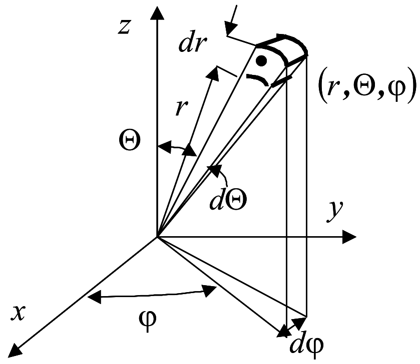

Fig. 3

In spherical coordinate system we have $T = T(r,\theta,\varphi)$ (Fig. 3),

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\Theta}\frac{\partial}{\partial \Theta}\left(\sin\Theta \frac{\partial}{\partial \Theta}\right) + \frac{1}{r^2 \sin\Theta}\frac{\partial^2}{\partial \varphi^2},$$

$$\nabla = \mathbf{i}_r \frac{\partial}{\partial r} + \mathbf{i}_\varphi \frac{1}{r}\frac{\partial}{\partial \varphi} + \mathbf{i}_\theta \frac{1}{r \sin\Theta}\frac{\partial}{\partial \Theta},$$

and

$$\frac{1}{a}\frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\Theta}\frac{\partial}{\partial \Theta}\left(\sin\Theta \frac{\partial T}{\partial \Theta}\right) + \frac{1}{r^2 \sin\Theta}\frac{\partial^2 T}{\partial \varphi^2} + \frac{q_V}{\lambda}; \qquad (7)$$

$$q_r = -\lambda \frac{\partial T}{\partial r}; q_\varphi = -\lambda \frac{1}{r}\frac{\partial T}{\partial \varphi}; q_\theta = -\lambda \frac{1}{r \sin\Theta}\frac{\partial T}{\partial \Theta}. \qquad (8)$$

The coordinates r, Θ, φ connects with the coordinates x, y, z by relations $x = r \sin\Theta \cos\varphi$, $y = r\sin\Theta \sin\varphi$, $z = r\cos\Theta$.

Thermal conduction equation for bodies of canonical forms

The various coordinate systems are convenient when it is necessary to find the temperature in bodies of canonical form: for parallelepiped, cylinder, and sphere (fig.4). The equation (1) describes the three-dimensional temperature distribution in parallelepiped. When the heating is carried out so that temperature depends on one coordinate only (Fig.4,a), the thermal conduction equation takes the form

$$a\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{q_V}{\lambda}.$$

In the case of long cylinder heated from lateral surface (Fig.4,b) we have from (5) the equation

$$a\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{q_V}{\lambda}.$$

In the case of the ball surround by homogeneous heated liquid we com from (7) to the equation.

$$\frac{1}{a}\frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{q_V}{\lambda}.$$

All three canonic equation can be written together

$$a\frac{\partial T}{\partial t}=\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n\frac{\partial T}{\partial r}\right)+\frac{q_V}{\lambda}, \tag{9}$$

where $n=0{,}1{,}2$ for plate, cylinder , and sphere.

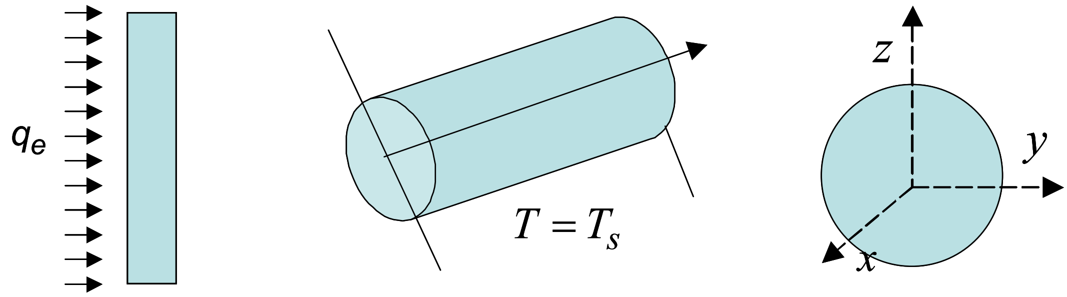

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Fig. 4

In dimensionless variables

$$\theta=\frac{T-T_0}{T_*-T_0};\ \tau=\frac{t}{t_*};\ \xi=\frac{r}{r_*},$$

where T_* is some specific temperature (for example, the temperature of washed liquid or hat wall); T_0 is initial temperature; r_* is the size specific for the problem (for example characteristic body size), the equation (9) takes the form

$$\frac{1}{Fo}\frac{\partial\theta}{\partial\tau}=\frac{1}{\xi^n}\frac{\partial}{\partial\xi}\left(\xi^n\frac{\partial\theta}{\partial\xi}\right)+\overline{q}_V, \tag{10}$$

where $Fo=\frac{\mathrm{a}t_*}{x_*^2}$ if Fourier number; $\overline{q}_V=\frac{q_V x_*^2}{\lambda(T_*-T_0)}$ is dimensionless density of internal heat sources. If we $\overline{q}_V=1$, we shall can the scale temperature

$$T_*=T_0+\frac{q_V}{\lambda}x_*^2.$$

Examples of stationary problems for cylindrical coordinate system

Consider the stationary process of thermal conduction in cylindrical wall with internal diameter $d_1=2r_1$ and external diameter $d_2=2r_2$ (Fig.5).

We assume that cylinder is long, so heat losses from its ends are not essential. Boundary conditions do not depend on coordinates φ and z. The thermal physical properties are constant.

The stationary temperature distribution follows from the equation

$$\frac{d^2T}{dx^2}+\frac{1}{r}\frac{dT}{dr}=0. \tag{11}$$

The solution of this equation has the form

$$T=C_1\,\mathbf{ln}\,r+C_2, \tag{12}$$

where C_1, C_2 - are integration constants.

The specific heat flux follows from (12)

$$q_r=-\lambda\frac{dT}{dr}=-\lambda\frac{C_1}{r}. \tag{13}$$

It diminishes in the direction of external surface

In the stationary conditions full neat flux passing though the section of cylindrical pipe of given length l and constant and equal to

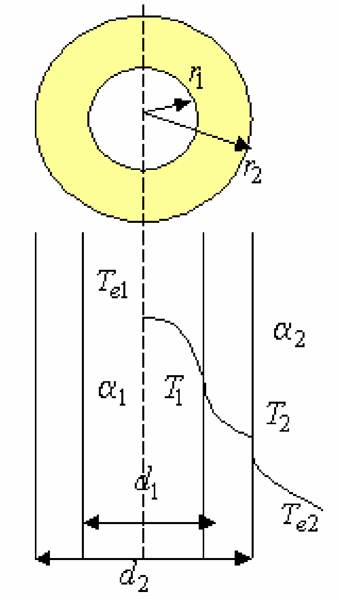

Fig. 5

$$Q = q_r F = q_r 2\pi r\, l = const \tag{14}$$

The second effect of cylindrical coordinate system – temperature changes nonlinearly along the pipe thickness oppositely to similar problem for Cartesian coordinate system.

The integration constants C_1 and C_2 can be found, if the boundary conditions will given.

The boundary conditions of first kind

Let the temperatures T_1 and T_2 are given on the cylinder surfaces $r = r_1$ and $r = r_2$. Then using (12) we shall find

$$T_1 = T(r_1) = C_1 \ln r_1 + C_2;$$

$$T_2 = T(r_2) = C_1 \ln r_2 + C_2.$$

Determining the constants, we come to the formulae

$$T = \frac{T_1 \ln(r_2/r) + T_2 \ln(r/r_1)}{\ln(r_2/r_1)}, \tag{15}$$

Therefore, the heat quantity passing through the section of length l during unit time is

$$Q = -\lambda \frac{dT}{dr} \cdot l \cdot 2\pi r \equiv \frac{\lambda \Delta T}{\ln(r_2/r_1)} \cdot 2\pi l, \text{ W}, \tag{16}$$

That is, it really does not depend on radius. S. т.е., действительно, не зависит от радиуса.

Linear heat flux has been used in technical calculations

$$q_l = \frac{Q}{l} = \frac{2\pi\lambda}{\ln(r_2/r_1)} \Delta T \tag{17}$$

Boundary conditions of third kind

When the temperatures of environments are given, and the wall temperatures are not known, we come to boundary conditions of third kind. It is necessary to give the neat emissions coefficients α_k also.

In this case, convective heat fluxes for the unit of the pipe length from external and internal surfaces follow from Newton laws and should be equal to linear heat flux due tu thermal conduction through cylindrical wall. We come to set of equations

$$q_l = \alpha_1 (T_{e1} - T_1) \cdot 2\pi r_1;$$

$$q_l = \frac{\lambda}{\ln(r_2/r_1)} (T_1 - T_2) \cdot 2\pi;$$

$$q_l = \alpha_2 (T_2 - T_{e2}) \cdot 2\pi r_2,$$

the solution of which gives linear heat flux density

$$q_l = \pi K_c (T_{e1} - T_{e2}), \tag{18}$$

where

$$K_c = \frac{1}{\dfrac{1}{2\alpha_1 r_1} + \dfrac{1}{2\lambda} \ln\left(\dfrac{r_2}{r_1}\right) + \dfrac{1}{2\alpha_2 r_2}} \text{ -}$$

heat transfer coefficient for cylindrical wall, W/(m K). Coefficient K_c equals numerically to the heat quantity transferring through the pipe wall of unit length during unit time, when temperature drop equals to 1 K.

The reverse value

$$K_c^{-1} = \frac{1}{2\alpha_1 r_1} + \frac{1}{2\lambda}\ln\left(\frac{r_2}{r_1}\right) + \frac{1}{2\alpha_2 r_2}$$

is called *full thermal resistance of the pipe*; называется *полным термическим сопротивлением трубы*; $1/(2\alpha_i r_i)$, $i = 1{,}2$ are thermal resistances of heat emission; $(1/2\lambda)\ln(r_2/r_1)$ is thermal resistance of thermal conduction.

It is follows from the sane equation set

$$T_1 = T_{e1} - \frac{K_c(T_{e1} - T_{e2})}{2\alpha_1 r_1};\ T_2 = \frac{K_c(T_{e1} - T_{e2})}{2\alpha_2 r_2} - T_{e2}.$$

That allows finding the temperature distribution.

Electrical analogy

The expression (16) can be written down in the form of Ohm law

$$Q = \frac{\Delta T}{\ln(r_2/r_1)/(2\pi l\lambda)},$$

where denominator presents the thermal resistance of hollow cylinder

$$R_T = \frac{\ln(r_2/r_1)}{2\pi\lambda l}.$$

The principles of successive and parallel combination of thermal resistance in circuit (chain) which are correct for plane wall can be use in the problem on hollow cylinder.

For example, let a liquid flows in pipe with isolation (Fig. 6). Here we can determine convective heat resistance of liquid

$$R_0 = \frac{1}{\alpha F} = \frac{1}{\alpha 2\pi r_1 l},$$

thermal resistance of hollow cylinder and isolation. We have successive combination of convective resistance of liquid with two conductive thermal resistances. If the temperatures of liquid and external surface are given we have

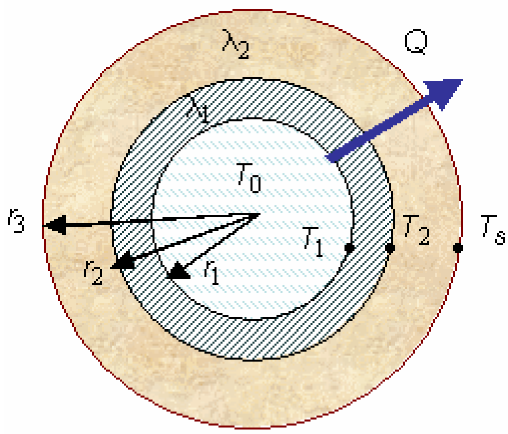

Fig.6. The cross-section of isolated pipe

$$Q = \left(\frac{\Delta T}{R}\right)_{full} = \frac{T_0 - T_s}{\frac{1}{2\pi\alpha_1 r_1 l} + \frac{1}{2\pi l\lambda_1}\ln\left(\frac{r_2}{r_1}\right) + \frac{1}{2\pi l\lambda_2}\ln\left(\frac{r_3}{r_2}\right)}. \qquad (19)$$

Thermal resistance in (19) is a sum of all listed thermal resistances.

The same principles are suitable for multilayer cylindrical wall.

Critical diameter of heat isolation

From above we know that radial heat flux in pipe inversely to logarithm of external radius; but flux emission from external surface is proportional this radius. That indicates that there is certain radius, when the heat losses are minimal.

If we increase the wall thickness due to the radius r_2 evaluation at the constant and small r_1, the action of logarithm item in heat flux will more strong. If r_1 is fixed, we have $q = q(r_2)$. Therefore, the flux will maximal when

$$\frac{dq}{dr_2} = -\frac{2\pi\Delta T\left(\frac{1}{\lambda_1 r_2} - \frac{1}{\alpha_2 r_2^2}\right)}{\left(\frac{1}{\alpha_1 r_1} + \frac{1}{\lambda_1}\mathbf{ln}\frac{r_2}{r_1} + \frac{1}{\alpha_2 r_2}\right)^2} = 0.$$

That is possible, when

$$(r_2)_* = \frac{\lambda_1}{\alpha_2}, \qquad (20)$$

and leads to the interested effect: heat loss can be diminished due to isolation thickness reducing.

That can be illustrated for two layer pipe, the full thermal resistance of which follows from formula

$$R_C = \frac{1}{K_C} = \frac{1}{\alpha_1 d_1} + \frac{1}{2\lambda_1}\mathbf{ln}\left(\frac{d_2}{d_1}\right) + \frac{1}{2\lambda_2}\mathbf{ln}\left(\frac{d_3}{d_2}\right) + \frac{1}{\alpha_2 d_3},$$

where $d_1 = 2r_1$; $d_1 = 2r_1$; $d_1 = 2r_1$; λ_1 is the heat thermal conductivity of pipe material, ; λ_2 is the heat thermal conductivity of isolation material.

The extreme condition for this function is

$$(R_C)_{d_3}' = \left(\frac{1}{K_C}\right)_{d_3}' = \frac{1}{2\lambda_2 d_3} - \frac{1}{\alpha_2 d_3^2} = 0.$$

Therefore the critical diameter of isolation does not depend on pipe diameter d_2

$$(d_3)_* = \frac{2\lambda_2}{\alpha_2}$$

and is determined by the value of thermal conductivity coefficient λ_2 and the value α_2.

If $d_2 > (d_3)_*$ and the heat exchange conditions are given, the pipe coatication diminishes the heat losses. If $d_2 < (d_3)_*$, then coating firstly leads to heat loss increase.

The solutions of the simplest problems in dimensionless variables

It is no difficult to find the solutions on temperature distributions for ball wall.

For boundary conditions of first kind the solution takes the form

$$T(r) = \frac{T_1\left(\frac{1}{r} - \frac{1}{r_2}\right) + T_2\left(\frac{1}{r_1} - \frac{1}{r}\right)}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}.$$

All obtained solutions can be presented together in dimensionless variables. In the variables

$$\theta = \frac{T - T_2}{T_1 - T_2}; \ \xi = \frac{r}{r_2}$$

the solutions will have following form

$$\theta_p = 1 - \xi, \ 0 \le \xi \le 1;$$

$$\theta_c = -\mathbf{ln}\,\xi / \mathbf{ln}\,\varepsilon, \ \varepsilon^{-1} \le \xi \le 1;$$

$$\theta_b = (1 - \xi)/(\varepsilon - 1)/\xi, \ \varepsilon^{-1} \le \xi \le 1,$$

where $\varepsilon = r_2 / r_1 > 1$.

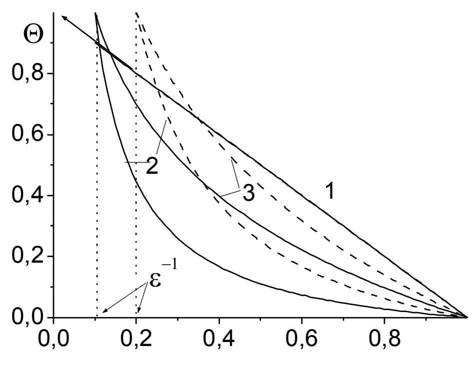

Fig. 7. *The temperature distribution in plane (1), cylindrical (2) and ball (3) wall. Solid lines - ε=10; dotted lines – ε=5*

The problems with volume heat sources

Along with the processes thermal conduction and heat exchange, the volume heat release is possible in substance connecting with some physical-chemical phenomena: condensation, Joule heating, nucleus and chemical reactions etc. From thermal physical point of view, we can characterize theses processes by the heat quantity releasing or absorbing in unit volume during unit time q_V, W/m^2. This characteristic is ***volume heat release intensity***. Assume this value is constant and does not depend on time and space coordinates. It is very simplified approach. We With meet more rigorous way of chemical reaction description in following lections.

Now we stop on the problem on the plate with volume source and given surface temperatures (Fig. 8). The formulation of the stationary problem has a view

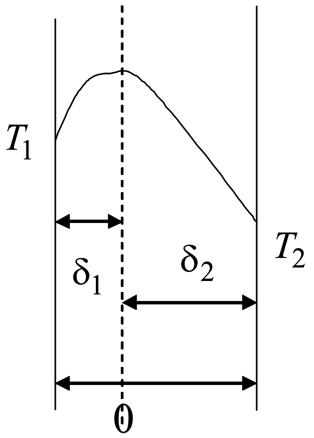

Fig. 8

$$\frac{d^2T}{dx^2}+\frac{q_V}{\lambda}=0$$

$$x=-\delta_1: \qquad T=T_1;$$

$$x=\delta_2: \qquad T=T_2,$$

The problem is solved by simple integration. But the solution way with temperature maximum extraction and its position is possible. The solution takes the form

$$T(x)=T_2+\frac{q_V}{2\lambda}\left\{\left[\frac{\delta}{2}\left(1+\frac{2\lambda(T_1-T_2)}{q_V\delta^2}\right)\right]^2-x^2\right\}$$

where δ is the sickness of the plate.

The maximum position from the right surface is

$$\delta_2=\frac{\delta}{2}\left(1+\frac{2\lambda(T_1-T_2)}{q_V\delta^2}\right).$$

Analogous problems take a place in cylindrical and spherical coordinate systems. Many applied problems are contained in the scientific and school literature.