Chapter 4

Transients

In this chapter we study circuits that contain sources, switches, resistances, inductances, and capacitances. The time-varying currents and voltages resulting from the sudden application of sources, possibly due to switching, are called **transients**.

In transient analysis, we start by writing circuit equations using concepts developed in Chapter 2, such as KCL, KVL, node-voltage analysis, and mesh-current analysis. Because the current–voltage relationships for inductances and capacitances involve integrals and derivatives, we obtain integrodifferential equations. These equations can be converted to pure differential equations by differentiating with respect to time. Thus the study of transients requires us to solve differential equations.

Study of this chapter will enable you to:

- Solve first-order RC or RL circuits.
- Understand the concepts of transient response and steady-state response.
- Relate the transient response of first-order circuits to the time constant.
- Solve RLC circuits in dc steady-state conditions.
- Solve second-order circuits.
- Relate the step response of a second-order system to the natural frequency and damping ratio.
- Write PSpice programs to perform transient analysis and use Probe to obtain plots of currents and voltages versus time.

4.1 FIRST-ORDER RC CIRCUITS

In this section we consider transients in circuits that contain independent dc sources, resistances, and a single capacitance.

4.1.1 Discharge of a Capacitance through a Resistance

As a first example, consider the circuit shown in Figure 4.1a. Prior to $t = 0$, the capacitor is charged to an initial voltage V_i. Then at $t = 0$ the switch closes and current flows through the resistor, discharging the capacitor.

Writing a current equation at the top node of the circuit after the switch is closed yields

$$C\frac{dv_C(t)}{dt} + \frac{v_C(t)}{R} = 0$$

Multiplying by the resistance yields

$$RC\frac{dv_C(t)}{dt} + v_C(t) = 0 \tag{4.1}$$

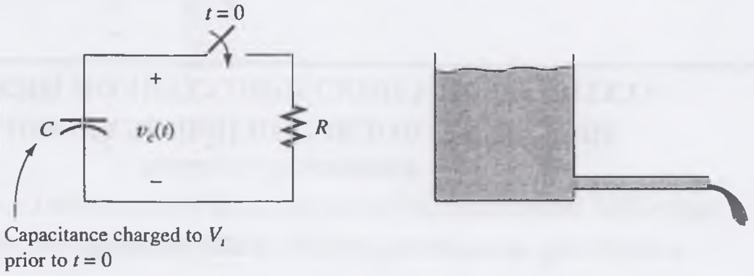
As expected, we have obtained a differential equation.

Equation 4.1 indicates that the solution for $v_C(t)$ must be a function that has the same form as its first derivative. Of course, a function with this property is an exponential. Thus we anticipate that the solution is of the form

$$v_C(t) = Ke^{st} \tag{4.2}$$

in which K and s are constants to be determined.

Figure 4.1 A capacitance discharging through a resistance and its fluid-flow analogy. The capacitor is charged to V_i prior to $t = 0$ (by circuitry that is not shown). At $t = 0$, the switch closes and the capacitor discharges through the resistor.



(a) Electrical circuit

(b) Fluid-flow analogy: a filled water tank discharging through a small pipe

Using Equation 4.2 to substitute for $v_C(t)$ in Equation 4.1, we have

$$RCKse^{st} + Ke^{st} = 0 \tag{4.3}$$

Solving for s, we obtain

$$s = \frac{-1}{RC} \tag{4.4}$$

Substituting this into Equation 4.2, we see that the solution is

$$v_C(t) = Ke^{-t/RC} \tag{4.5}$$

Referring to Figure 4.1a, we reason that the voltage across the capacitor cannot change instantaneously when the switch closes. This is because the current through the capacitance is $i_C(t) = C\,dv_C/dt$. In order for the voltage to change instantaneously, the current would have to be infinite. Since the voltage is finite, the current in the resistance must be finite, and we conclude that the voltage across the capacitor must be continuous. Thus we write

$$v_C(0+) = V_i \tag{4.6}$$

in which $v_C(0+)$ represents the voltage immediately after the switch closes. Substituting into Equation 4.5, we have

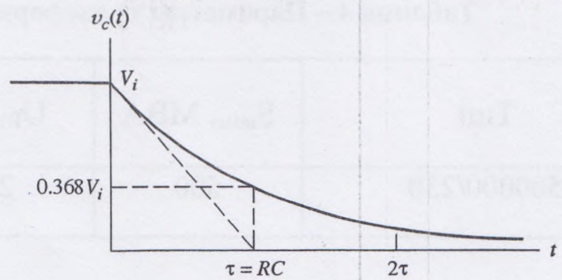
$$v_C(0+) = V_i = Ke^0 = K \tag{4.7}$$

Thus we conclude that the constant K equals the initial voltage across the capacitor. Finally, the solution for the voltage is

$$v_C(t) = V_i e^{-t/RC} \tag{4.8}$$

A plot of the voltage is shown in Figure 4.2. Notice that the capacitor voltage decays exponentially to zero.

Figure 4.2 Voltage versus time for the circuit of Figure 4.1a. When the switch is closed, the voltage across the capacitor decays exponentially to zero. At one time constant, the voltage is equal to 36.8% of its initial value.



The time interval

$$\tau = RC \tag{4.9}$$

is called the **time constant** of the circuit. In one time constant the voltage decays by the factor $e^{-1} \simeq 0.368$. After about five time constants the voltage remaining on the capacitor is negligible compared to the initial value.

An analogous fluid-flow system is shown in Figure 4.1b. The tank initially filled with water is analogous to the charged capacitor. Furthermore, the small pipe is analogous to the resistor. At first, when the tank is full, the flow is large and the water level drops fast. As the tank empties, the flow decreases.

We frequently apply RC circuits in timing circuits. For example, suppose that when a garage door opens or closes, a light is to be turned on and remain on for 30 s. To achieve this objective, we could design a circuit consisting of (1) a capacitor that is charged to an initial voltage V_i while the door opener is energized, (2) a resistor through which the capacitor discharges, and (3) a sensing circuit that keeps the light on as long as the capacitor voltage is larger than $0.368V_i$. If we choose the time constant $\tau = RC$ to be 30 s, the desired operation is achieved. In later chapters we learn more about the design of this type of circuit.

4.1.2 Charging a Capacitance from a DC Source through a Resistance

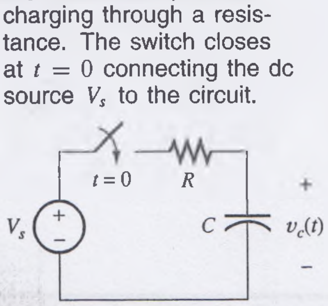
Next consider the circuit shown in Figure 4.3. The source voltage V_s is constant—in other words, we have a dc source. The source is connected to the RC circuit by a switch that closes at $t = 0$. We assume that the initial voltage across the capacitor just before the switch closes is $v_C(0-) = 0$. Let us solve for the voltage across the capacitor as a function of time.

We start by writing a current equation at the node that joins the resistor and the capacitor. This yields

$$C\frac{dv_C(t)}{dt} + \frac{v_C(t) - V_s}{R} = 0 \tag{4.10}$$

The first term on the left-hand side is the current referenced downward through the capacitor. The second term is the current referenced toward the left through the resistor. KCL requires that the currents leaving the node sum to zero.

Figure 4.3 Capacitance charging through a resistance. The switch closes at $t = 0$ connecting the dc source V_s to the circuit.



Rearranging Equation 4.10, we obtain

$$RC\frac{dv_C(t)}{dt} + v_C(t) = V_s \tag{4.11}$$

As expected, we have obtained a linear first-order differential equation with constant coefficients. As in the previous circuit, the voltage across the capacitance cannot change instantaneously. Thus we have

$$v_C(0+) = v_C(0-) = 0 \tag{4.12}$$

Now we need to find a solution for $v_C(t)$ that (1) satisfies Equation 4.11 and (2) matches the initial conditions of the circuit stated in Equation 4.12. Notice that Equation 4.11 is the same as Equation 4.1 except for the constant on the right-hand side. Thus we expect the solution to be the same as for Equation 4.1 except for an added constant term. Thus we are led to try the solution

$$v_C(t) = K_1 + K_2e^{st} \tag{4.13}$$

in which K_1, K_2, and s are constants to be determined.

If we use Equation 4.13 to substitute for $v_C(t)$ in Equation 4.11, we obtain

$$(1 + RCs)K_2e^{st} + K_1 = V_s \tag{4.14}$$

For equality, the coefficient of e^{st} must be zero. This leads to

$$s = \frac{-1}{RC} \tag{4.15}$$

From Equation 4.14, we also have

$$K_1 = V_s \tag{4.16}$$

Using Equations 4.15 and 4.16 to substitute into Equation 4.13, we obtain

$$v_C(t) = V_s + K_2e^{-t/RC} \tag{4.17}$$

in which K_2 remains to be determined.

Now we use the initial condition (Equation 4.12) to find K_2. We have

$$v_C(0+) = 0 = V_s + K_2e^0 = V_s + K_2 \tag{4.18}$$

from which we find $K_2 = -V_s$. Finally, substituting into Equation 4.17, we obtain the solution:

$$v_C(t) = V_s - V_se^{-t/RC} \tag{4.19}$$

The second term on the right-hand side is called the **transient response**, which eventually decays to negligible values. The first term on the right-hand side is the **steady-state response**, which persists after the transient has decayed.

Here again, the product of the resistance and capacitance has units of seconds and is called the time constant $\tau = RC$. Thus the solution can be written as

$$v_C(t) = V_s - V_s e^{-t/\tau} \tag{4.20}$$

A plot of $v_C(t)$ is shown in Figure 4.4. Notice that $v_C(t)$ starts at 0 and approaches the final value V_s asymptotically as t becomes large. After one time constant, $v_C(t)$ has reached 63.2% of its final value. For practical purposes, $v_C(t)$ is equal to its final value V_s after about five time constants. Then we say that the circuit has reached steady state.

It can be shown that if the initial slope of v_C is extended, it intersects the final value at one time constant as shown in Figure 4.4.

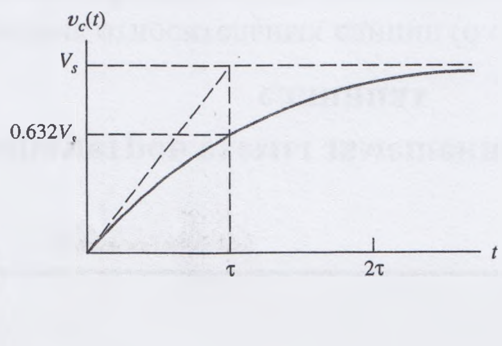
We have seen in this section that several time constants are needed to charge or discharge a capacitance. This is the main limitation on the speed at which digital computers can process data. In a typical computer, information is represented by voltages that nominally assume values of either +5 or 0 V, depending on the data represented. When the data change, the voltages must change. It is impossible to build circuits that do not have some capacitance that is charged or discharged when voltages change in value. Furthermore, the circuits always have nonzero resistances that limit the currents available for charging or discharging the capacitances. Therefore, a nonzero time constant is associated with each circuit in the computer, limiting its speed. We will learn more about digital computer circuits in later chapters.

EXERCISE 4.1 Suppose that $R = 5000\ \Omega$ and $C = 1\mu$F in the circuit of Figure 4.1a. Find the time at which the voltage across the capacitor reaches 1% of its initial value.

Ans. $t = -5\ln(0.01)$ ms $\simeq 23$ ms.

EXERCISE 4.2 Show that if the initial slope of $v_C(t)$ is extended, it intersects the final value at one time constant, as shown in Figure 4.4. [The expression for $v_C(t)$ is given in Equation 4.20.]

Figure 4.4 The charging transient for the RC circuit of Figure 4.3.



4.2 DC STEADY STATE

The transient terms in the expressions for currents and voltages in RLC circuits decay to zero with time. (An exception is LC circuits having no resistance.) For dc sources, the steady-state currents and voltages are also constant.

Consider the equation for current through a capacitance:

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

If the voltage $v_C(t)$ is constant, the current is zero. In other words, the capacitance behaves as an open circuit. Thus we conclude that *for steady-state conditions with dc sources, capacitances behave as open circuits.*

Similarly, for an inductance, we have

$$v_L(t) = L\frac{di_L(t)}{dt}$$

When the current is constant, the voltage is zero. Thus we conclude that *for steady-state conditions with dc sources, inductances behave as short circuits.*

These observations give us another approach to finding the steady-state solutions to circuit equations for RLC circuits with constant sources. First, we replace the capacitors by open circuits and the inductors by short circuits. The circuit then consists of dc sources and resistances. We solve the equivalent circuit for the steady-state currents and voltages.

Example 4.1

Find v_x and i_x for the circuit shown in Figure 4.5a for $t \gg 0$.

Solution

After the switch has been closed a long time, we expect the transient response to have decayed to zero. Then the circuit is operating in dc steady-state conditions. We start our analysis by replacing the inductor by a short circuit and the capacitor by an open circuit. The equivalent circuit is shown in Figure 4.5b.

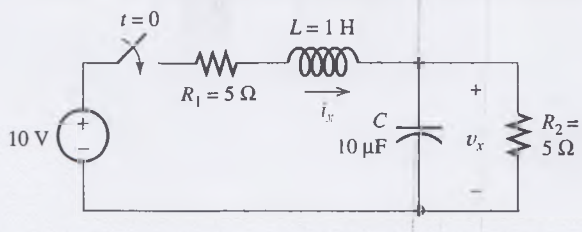
This resistive circuit is readily solved. The resistances R_1 and R_2 are in series. Thus we have

$$i_x = \frac{10}{R_1 + R_2} = 1\text{ A}$$

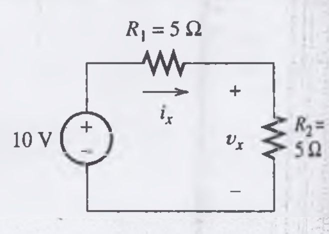
and

$$v_x = R_2 i_x = 5\text{ V}$$

Figure 4.5 The circuit and its dc steady-state equivalent for Example 4.1.



(a) Original circuit



(b) Equivalent circuit for steady state

Sometimes we are only interested in the steady-state operation of circuits with dc sources. For example, in analyzing the headlight circuits in an automobile, we are concerned primarily with steady state. On the other hand, we must consider transients in analyzing the operation of the ignition system.

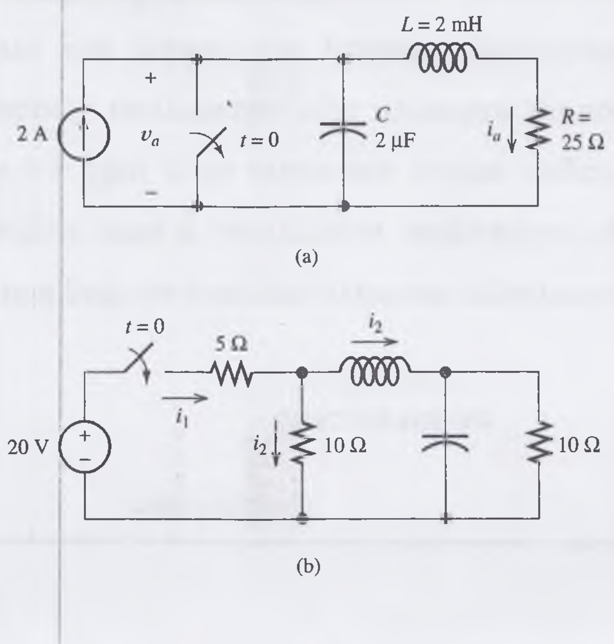
In other applications, we are interested in steady-state conditions with sinusoidal ac sources. For sinusoidal sources the steady-state currents and voltages are also sinusoidal. In Chapter 5 we study a method for solving sinusoidal steady-state circuits that is similar to the method we have presented here for dc steady state. Instead of short and open circuits, we will replace inductances and capacitances by impedances, which are like resistances except that impedances can have imaginary values.

EXERCISE 4.3 Solve for the steady-state values of the labeled currents and voltages for the circuits shown in Figure 4.6.
Ans. **(a)** $v_a = 50$ V, $i_a = 2$ A; **(b)** $i_1 = 2$ A, $i_2 = 1$ A, $i_3 = 1$ A.

4.3 *RL* CIRCUITS

In this section we consider circuits consisting of dc sources, resistances, and a single inductance. The methods and solutions are very similar to those we studied for *RC* circuits in Section 4.1.

Figure 4.6 Circuits for Exercise 4.3.



Example 4.2 Consider the circuit shown in Figure 4.7. Find the current $i(t)$ and the voltage $v(t)$.

Solution First we find the current $i(t)$. Of course, prior to $t = 0$, the switch is open and the current is zero.

$$i(t) = 0 \qquad \text{for } t < 0 \tag{4.21}$$

After the switch is closed, the current increases in value eventually reaching a steady-state value.

Writing a KVL equation around the loop, we have

$$Ri(t) + L\frac{di}{dt} = V_s \tag{4.22}$$

This is very similar to Equation 4.11, and we are therefore led to try a solution of the same form as that given by Equation 4.13. Thus our trial solution is

$$i(t) = K_1 + K_2 e^{st} \tag{4.23}$$

in which K_1, K_2, and s are constants that need to be determined. Following the procedure used in Section 4.1, we substitute the trial solution into the differential equation, resulting in

$$RK_1 + (RK_2 + sLK_2)e^{st} = V_s \tag{4.24}$$

from which we obtain

$$K_1 = \frac{V_s}{R} = 2 \tag{4.25}$$

and

$$s = \frac{-R}{L} \tag{4.26}$$

Substituting these values into Equation 4.23 results in

$$i(t) = 2 + K_2 e^{-tR/L} \tag{4.27}$$

Next we use the initial conditions to determine the value of K_2. The current in the inductor is zero prior to $t = 0$ because the switch is open. The applied voltage is finite, and the inductor current must be continuous (because $v_L = L\,di/dt$).

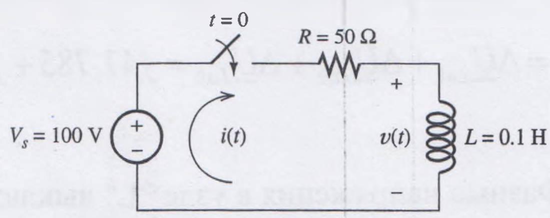


Figure 4.7 The circuit analyzed in Example 4.2.

Thus, immediately after the switch is closed, the current must be zero. Thus we have

$$i(0+) = 0 = 2 + Ke^0 = 2 + K_2 \tag{4.28}$$

Solving, we find that $K_2 = -2$.

Substituting into Equation 4.27, the solution for the current is

$$i(t) = 2 - 2e^{-t/\tau} \qquad \text{for } t > 0 \tag{4.29}$$

in which the time constant is given by

$$\tau = \frac{L}{R} \tag{4.30}$$

A plot of the current versus time is shown in Figure 4.8a. Notice that the current increases from zero to the steady-state value of 2 A. After five time constants the current is within 99% of the final value. As a check we verify that the steady-state current is 2 A. (As we saw in Section 4.2, this value can be obtained directly by treating the inductor as a short circuit.)

Now we consider the voltage $v(t)$. Prior to $t = 0$ with the switch open the voltage is zero.

$$v(t) = 0 \qquad \text{for } t < 0 \tag{4.31}$$

After $t = 0$, $v(t)$ is equal to the source voltage minus the drop across R. Thus we have

$$v(t) = 100 - 50i(t) \qquad \text{for } t > 0 \tag{4.32}$$

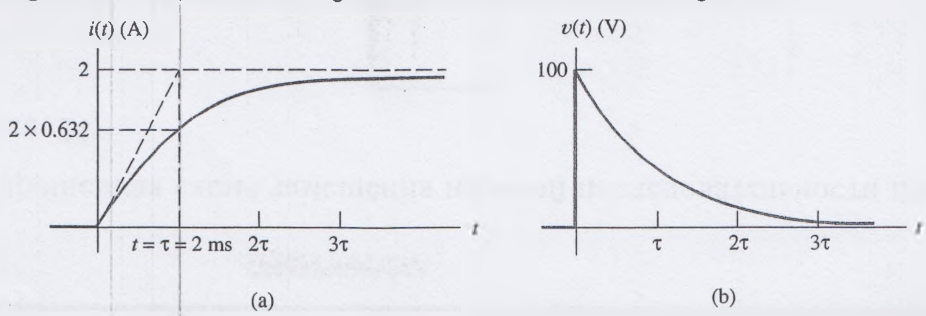
Substituting the expression found earlier for $i(t)$, we obtain

$$v(t) = 100e^{-t/\tau} \tag{4.33}$$

A plot of $v(t)$ is shown in Figure 4.8b.

At $t = 0$, the voltage across the inductor jumps from 0 to 100 V. As the current gradually increases, the drop across the resistor increases, and the voltage across the inductor falls. In steady state, we have $v(t) = 0$ because the inductor behaves as a short circuit. ■

Figure 4.8 Current and voltage versus time for the circuit of Figure 4.7.



Example 4.3 Consider the circuit shown in Figure 4.9 in which V_s is a dc source. Assume that the circuit is in steady state prior to $t = 0$. Find expressions for the current $i(t)$ and the voltage $v(t)$.

Solution Prior to $t = 0$, the inductor behaves as a short circuit. Thus we have

$$v(t) = 0 \qquad \text{for } t < 0$$

and

$$i(t) = \frac{V_s}{R_1} \qquad \text{for } t < 0$$

Before the switch opens, current circulates clockwise through V_s, R_1, and the inductance. When the switch opens, current continues to flow through the inductance, but the return path is through R_2. Then a voltage appears across R_2 and the inductance, causing the current to decay.

Since there are no sources driving the circuit after the switch opens, the steady-state solution is zero for $t > 0$. Thus the solution for $i(t)$ is given by

$$i(t) = Ke^{-t/\tau} \qquad \text{for } t > 0 \tag{4.34}$$

in which the time constant is

$$\tau = \frac{L}{R_2} \tag{4.35}$$

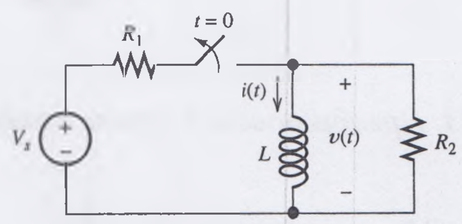
Unless an infinite voltage appears across the inductance, the current must be continuous. Recall that prior to $t = 0$, $i(t) = V_s/R_1$. Thus just after the switch opens, we have

$$i(0+) = \frac{V_s}{R_1} = Ke^{-0} = K$$

Substituting the value of K into Equation 4.34, the current is

$$i(t) = \frac{V_s}{R_1}e^{-t/\tau} \qquad \text{for } t > 0 \tag{4.36}$$

Figure 4.9 The circuit analyzed in Example 4.3.



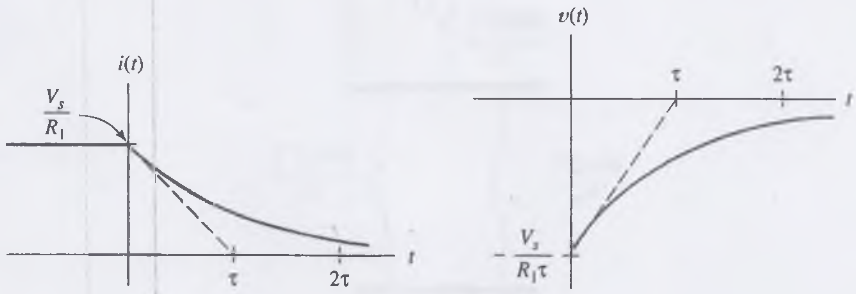


Figure 4.10 The current and voltage for the circuit of Figure 4.9.

The voltage is given by

$$v(t) = L\frac{di(t)}{dt}$$

$$= 0 \qquad \text{for } t < 0$$

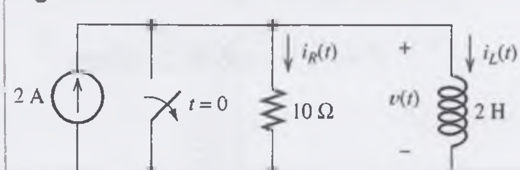
$$= -\frac{V_s}{R_1\tau}e^{-t/\tau} \qquad \text{for } t > 0$$

Plots of the voltage and current are shown in Figure 4.10. ■

EXERCISE 4.4 For the circuit of Example 4.3, assume that $V_s = 15$ V, $R_1 = 10\Omega$, $R_2 = 100\Omega$, and $L = 0.1$ H. **(a)** What is the value of the time constant (after the switch opens)? **(b)** What is the maximum magnitude of $v(t)$? **(c)** How does the maximum magnitude of $v(t)$ compare to the source voltage? **(d)** Find the time t at which $v(t)$ is one-half of its value immediately after the switch opens.
Ans. **(a)** $\tau = 1$ ms; **(b)** $|v(t)|_{max} = 150$ V; **(c)** the maximum magnitude of $v(t)$ is 10 times the value of V_s; **(d)** $t = \tau\ln(2) = 0.693$ ms.

EXERCISE 4.5 Consider the circuit shown in Figure 4.11, in which the switch opens at $t = 0$. Find expressions for $v(t)$, $i_R(t)$, and $i_L(t)$ for $t > 0$. Assume that $i_L(t)$ is zero before the switch opens.
Ans. $v(t) = 20e^{-t/0.2}$, $i_R(t) = 2e^{-t/0.2}$, $i_L(t) = 2 - 2e^{-t/0.2}$.

Figure 4.11 The circuit for Exercise 4.5.



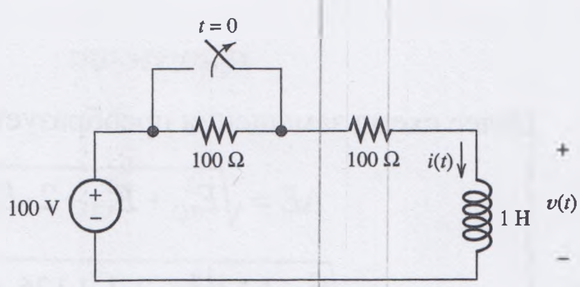


Figure 4.12 The circuit for Exercise 4.6.

EXERCISE 4.6 Consider the circuit shown in Figure 4.12. Assume that the switch has been closed for a very long time prior to $t = 0$. Find expressions for $i(t)$ and $v(t)$.

Ans.

$$\begin{aligned} i(t) &= 1.0 \quad \text{for } t < 0 \\ &= 0.5 + 0.5e^{-t/\tau} \quad \text{for } t > 0 \\ v(t) &= 0 \quad \text{for } t < 0 \\ &= -100e^{-t/\tau} \quad \text{for } t > 0 \end{aligned}$$

where the time constant is $\tau = 5$ ms.

4.4 *RC* AND *RL* CIRCUITS WITH GENERAL SOURCES

Now that we have gained some familiarity with RL and RC circuits, we discuss their solution in general. In this section we treat circuits that contain one energy-storage element, either an inductance or a capacitance.

Consider the circuit shown in Figure 4.13a. The circuit inside the box can be any combination of resistances and sources. The single inductance L is shown explicitly. Recall that we can find a Thévenin equivalent for circuits consisting of sources and resistances. The Thévenin equivalent is an independent voltage source $v_t(t)$ in series with the Thévenin resistance R. Thus any circuit composed of sources, resistances, and one inductance has the equivalent circuit shown in Figure 4.13b. (Of course, we could reduce any circuit containing sources, resistances, and a single capacitance in a similar fashion.)

Writing a KVL equation for Figure 4.13b, we obtain

$$L\frac{di(t)}{dt} + Ri(t) = v_t(t) \tag{4.37}$$

If we divide through by the resistance R, we have

$$\frac{L}{R}\frac{di(t)}{dt} + i(t) = \frac{v_t(t)}{R} \tag{4.38}$$

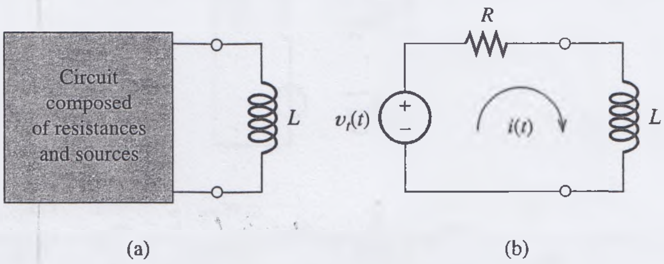


Figure 4.13 A circuit consisting of sources, resistances, and one inductance has an equivalent circuit consisting of a voltage source and a resistance in series with the inductance.

In general, the equation for any circuit containing one inductance or one capacitance can be put into the form

$$\tau \frac{dx(t)}{dt} + x(t) = f(t) \tag{4.39}$$

Then we need to find solutions to Equation 4.39 that are consistent with the initial conditions (such as the initial current in the inductance).

The constant τ (which turns out to be the time constant) is a function of only the resistances and the inductance (or capacitance). The sources result in the term $f(t)$, which is called the **forcing function**. If we have a circuit without sources (such as Figure 4.1), the forcing function is zero. For dc sources, the forcing function is constant.

Equation 4.39 is called a first-order differential equation because the highest-order derivative is first order. It is a linear equation because it does not involve powers or other nonlinear functions of $x(t)$ or its derivatives. Thus to solve an RL (or RC) circuit, we must find the general solution of a linear first-order differential equation with constant coefficients.

4.4.1 Solution of the Differential Equation

An important result in differential equations states that the general solution to Equation 4.39 consists of two parts. The first part is called the **particular solution** $x_p(t)$ and is any expression that satisfies Equation 4.39. Thus

$$\tau \frac{dx_p(t)}{dt} + x_p(t) = f(t) \tag{4.40}$$

The particular solution is also called the **forced response** because it depends on the forcing function (which in turn is due to the independent sources).

(Even though the particular solution satisfies the differential equation, it may not be consistent with the initial conditions, such as the initial voltage on a capacitance or current through an inductance. By adding another term, known as the complementary solution, we obtain a general solution that satisfies both the differential equation and meets the initial conditions.)

For the forcing functions that we will encounter, we can select the form of the particular solution by inspection. For example, consider the forcing function

$$f(t) = 10\cos(200t)$$

Because the derivatives of sin and cos functions are also sin and cos functions, we would try a particular solution of the form

$$x_p(t) = A\cos(200t) + B\sin(200t)$$

where A and B are constants that must be determined. We find these constants by substituting the proposed solution into the differential equation. This leads to equations that can be solved for A and B. (In Chapter 5 we study shortcut methods for solving for the forced response of circuits with sinusoidal sources.)

The second part of the general solution is called the **complementary solution** $x_c(t)$ and is the solution of the **homogeneous equation**:

$$\tau \frac{dx_c(t)}{dt} + x_c(t) = 0 \tag{4.41}$$

We obtain the homogeneous equation, by setting the forcing function to zero. Thus the form of the complementary solution does not depend on the sources. It is also called the **natural response** because it depends on the passive circuit elements. The complementary solution must be added to the particular solution in order to obtain a general solution that matches the initial values of the currents and voltages.

We can rearrange the homogeneous equation into this form:

$$\frac{dx_c(t)/dt}{x_c(t)} = \frac{-1}{\tau} \tag{4.42}$$

Integrating both sides of Equation 4.42, we have

$$\ln[x_c(t)] = \frac{-t}{\tau} + c \tag{4.43}$$

in which c is the constant of integration. Equation 4.43 is equivalent to

$$x_c(t) = e^{(-t/\tau + c)} = e^c e^{-t/\tau} = Ke^{-t/\tau} \tag{4.44}$$

in which we have defined $K = e^c$.

4.4.2 Step-by-Step Solution

Next we summarize an approach to solving circuits containing a resistance, a source, and an inductance (or a capacitance).

1. Write the circuit equation and reduce it to a first-order differential equation.
2. Find a particular solution. The details of this step depend on the form of the forcing function. We illustrate several types of forcing functions in examples, exercises, and problems.
3. Obtain the complete solution by adding the particular solution to the complementary solution given by Equation 4.44, which contains the arbitrary constant K.
4. Use initial conditions to find the value of K.

We illustrate this procedure with an example.

Example 4.4 Solve for the current in the circuit shown in Figure 4.14. The capacitor is initially charged so that $v_C(0+) = 1$ V.

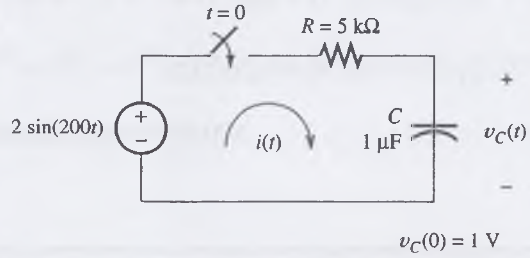
Solution First we write a voltage equation for $t > 0$. Traveling clockwise and summing voltages, we obtain

$$Ri(t) + \frac{1}{C}\int_0^t i(t)\,dt + v_C(0) - 2\sin(200t) = 0$$

We convert this to a differential equation by taking the derivative of each term. Of course, the derivative of the integral is simply the integrand. Because $v_C(0)$ is a constant, its derivative is zero. Thus we have

$$R\frac{di(t)}{dt} + \frac{1}{C}i(t) = 400\cos(200t) \tag{4.45}$$

Figure 4.14 A first-order RC circuit with a sinusoidal source. See Example 4.4.



Multiplying by C, we have

$$RC\frac{di(t)}{dt} + i(t) = 400C\cos(200t) \tag{4.46}$$

Substituting values for R and C, we have

$$5 \times 10^{-3}\frac{di(t)}{dt} + i(t) = 400 \times 10^{-6}\cos(200t) \tag{4.47}$$

The second step is to find a particular solution $i_p(t)$. Often, we start by guessing at the form of $i_p(t)$, possibly including some unknown constants. Then we substitute our guess into the differential equation and solve for the constants. In the present case, since the derivatives of $\sin(200t)$ and $\cos(200t)$ are $200\cos(200t)$ and $-200\sin(200t)$, respectively, we try a particular solution of the form

$$i_p(t) = A\cos(200t) + B\sin(200t) \tag{4.48}$$

where A and B are constants to be determined so that i_p is indeed a solution to Equation 4.47.

Substituting the proposed solution into Equation 4.47, we obtain

$$\begin{aligned} &-A\sin(200t) + B\cos(200t) + A\cos(200t) + B\sin(200t) \\ &= 400 \times 10^{-6}\cos(200t) \end{aligned}$$

However, the left-hand side of this equation is required to be identical to the right-hand side. Equating the coefficients of the sine functions, we have

$$-A + B = 0 \tag{4.49}$$

Equating the coefficients of the cosine functions, we have

$$B + A = 400 \times 10^{-6} \tag{4.50}$$

These equations can be readily solved, yielding

$$A = 200 \times 10^{-6} = 200\ \mu\text{A}$$

and

$$B = 200 \times 10^{-6} = 200\ \mu\text{A}$$

Substituting these values into Equation 4.48, we obtain the particular solution.

$$i_p(t) = 200\cos(200t) + 200\sin(200t)\ \mu\text{A} \tag{4.51}$$

This can also be written as

$$i_p(t) = 200\sqrt{2}\cos(200t - 45°)$$

(In Chapter 5 we will learn shortcut methods of combining sin and cos functions.)

We obtain the homogeneous equation by substituting 0 for the forcing function in Equation 4.46. Thus we have

$$RC\frac{di(t)}{dt} + i(t) = 0 \tag{4.52}$$

The complementary solution is

$$i_c(t) = Ke^{-t/RC} = Ke^{-t/\tau} \tag{4.53}$$

Adding the particular solution and the complementary solution, we obtain the general solution:

$$i(t) = 200\cos(200t) + 200\sin(200t) + Ke^{-t/RC}\ \mu\text{A} \tag{4.54}$$

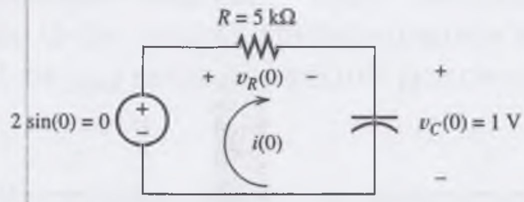
Finally, we determine the value of the constant K by using the initial conditions. The voltages and currents immediately after the switch closes are shown in Figure 4.15. The source voltage is 0 V and the voltage across the capacitor is $v_C(0+) = 1$. Consequently, the voltage across the resistor must be $v_R(0+) = -1$ V. Thus we have

$$i(0+) = \frac{v_R(0+)}{R} = \frac{-1}{5000} = -200\ \mu\text{A}$$

Substituting $t = 0$ into Equation 4.54, we have

$$i(0+) = -200 = 200 + K\ \mu\text{A} \tag{4.55}$$

Figure 4.15 The voltages and currents for the circuit of Figure 4.14 immediately after the switch closes.



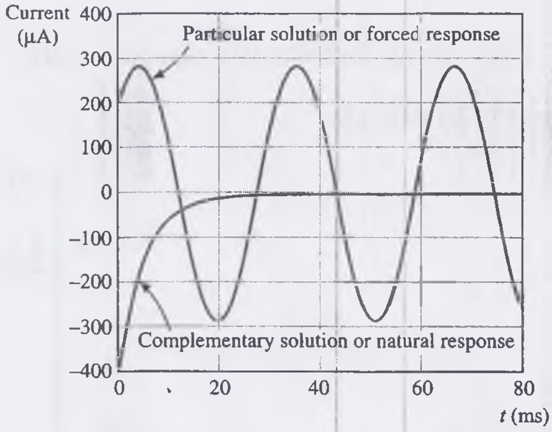


Figure 4.16 The complementary solution and the particular solution for Example 4.4.

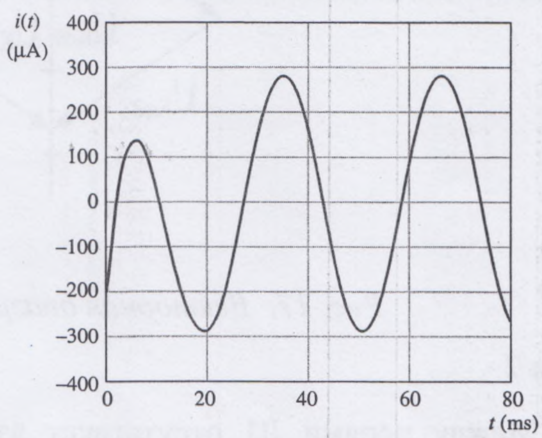
Solving, we find that $K = -400\ \mu\text{A}$. Substituting this into Equation 4.54, we have the solution:

$$i(t) = 200\cos(200t) + 200\sin(200t) - 400e^{-t/RC}\ \mu\text{A} \tag{4.56}$$

Plots of the particular solution and of the complementary solution are shown in Figure 4.16. The time constant for this circuit is $\tau = RC = 5$ ms. Notice that the natural response decays to negligible values in about 25 ms. As expected, the natural response has decayed in about five time constants. Furthermore, notice that for a sinusoidal forcing function, the forced response is also sinusoidal and persists after the natural response has decayed.

A plot of the complete solution is shown in Figure 4.17. ■

Figure 4.17 The complete solution for Example 4.4.



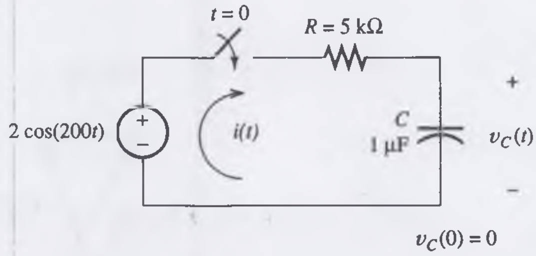


Figure 4.18 The circuit for Exercise 4.7.

EXERCISE 4.7 Repeat Example 4.4 if the source voltage is changed to $2\cos(200t)$ and the initial voltage on the capacitor is $v_C(0) = 0$. The circuit with these changes is shown in Figure 4.18.

Ans. $i(t) = -200\sin(200t) + 200\cos(200t) + 200e^{-t/RC}\ \mu\text{A}$, in which $\tau = RC = 5$ ms.

EXERCISE 4.8 Solve for the current in the circuit shown in Figure 4.19 after the switch closes. [*Hint:* Try a particular solution of the form $i_p(t) = Ae^{-t}$.]

Ans. $i(t) = 20e^{-t} - 15e^{-t/2}\ \mu\text{A}$.

4.5 SECOND-ORDER CIRCUITS

In this section we consider circuits that contain two energy-storage elements. In particular, we look at circuits that have an inductance and a capacitance either in series or in parallel.

4.5.1 Differential Equation

To derive the general form of the equations that we encounter in circuits with two energy-storage elements, consider the series circuit shown in Figure 4.20a. Writing a KVL equation, we have

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(t)dt + v_C(0) = v_s(t) \tag{4.57}$$

Figure 4.19 The circuit for Exercise 4.8.