NATIONAL RESEARCH TOMSK POLYTECHNIC UNIVERSITY
NATIONAL RESEARCH TOMSK STATE UNIVERSITY
MOSCOW INSTITUTE OF ELECTRONICS AND MATHEMATICS

V.G. Bagrov, V.V. Belov,
V.N. Zadorozhnyi, A.Yu. Trifonov

METHODS OF MATHEMATICAL PHYSICS

Special Functions.
Equations of Mathematical Physics

It is recommended for publishing as a study aid
by the Editorial Board of Tomsk Polytechnic University

Tomsk Polytechnic University Publishing House
2012

UDC 53:51(075.8)
BBC 22.311я73
M54

Bagrov V.G.

M54 Methods of Mathematical Physics. Special Functions. Equations of Mathematical Physics: study aid / V.G. Bagrov, V.V. Belov, V.N. Zadorozhnyi, A.Yu. Trifonov; Tomsk Polytechnic University, Tomsk State University, Moscow Institute of Electronics and Mathematics. – Tomsk: TPU Publishing House, 2012. – 257 p.

This textbook presents the theory of special functions and the methods for solving integral equations and first order and second order partial differential equations. The theoretical material is consistent to the existing courses of higher mathematics for engineering and physical specialties of universities.

The textbook can be useful to the graduate students, magisters, and post-graduates specializing in theoretical and mathematical physics.

The textbook is intended for students and post-graduates of physical and engineering departments.

UDC 53:51(075.8)
BBC 22.311я73

Reviewer

Professor, RAS Full Member,
Faculty of Mathematics of the Physical Department
of Moscow State University

V.P. Maslov

The work was partially supported by the Russian Federal program «Kadry» under contracts P691, 16.740.11.0469 and program «Nauka» No 1.604.2011

Contents

Preface

This section of the course deals with the properties of special functions arising in solving problems of mathematical physics. These solutions, even for comparatively simple physical problems which admit exact analytical solutions, often cannot be expressed in terms of elementary functions. Therefore special functions form the basis for constructing solutions to problems of mathematical physics, and this base is continuously extended in studying new, not treated earlier physical and mathematical problems. The main learning aims here are the following: The first one is to make the reader acquainted with the most usable special functions (Bessel functions, classical orthogonal polynomials, and the like). The second one is, based on the knowledge of various properties of specific functions, to gain an understanding of common methods and techniques suitable for investigating special functions which are not involved in the given course. The third one is to make the learner skilled in utilizing common methods in specific cases through the selection of proper problems. These aims just determine the content of this section of the textbook.

PART III

Special Functions

CHAPTER 1

The Sturm–Liouville Problem for Ordinary Differential Equations

In a narrow sense, by special functions are meant the functions that arise in solving partial differential equations, for instance, by the method of separation of variables. In particular, when using the method of separation of variables in cylindrical and spherical coordinates, we arrive at cylindrical and spherical functions.

A characteristic feature of these functions is that each of them is as a rule a solution of the equation

$$\frac{d}{dt}\Big[k(t)\frac{dy}{dt}\Big] - q(t)y = 0$$

with singular points, such that the coefficient $k(t)$ vanishes at one or several points of the interval of variation of the variable t. The solution of equations of this type has a number of specific properties some of which are considered in this chapter.

In a broader sense, by special functions in mathematical physics is meant a family of separate classes of nonelementary functions arising when solving both theoretical and applied problems in various fields of mathematics, physics, and engineering.

1 Boundary value problems for ordinary differential equations

Consider the nonhomogeneous linear differential equation

$$x'' + p(t)x' + q(t)x = f(t), \tag{1.1}$$

where $p(t), q(t)$ are functions continuous on the interval $[a, b]$. From the course of higher mathematics it is known that the general solution of Eq. (1.1) is

$$x(t) = C_1x_1(t) + C_2x_2(t) + \tilde{x}(t), \tag{1.2}$$

where C_1 and C_2 are arbitrary constants and $x_1(t)$ and $x_2(t)$ are solutions of Eq. (1.1) defined on the interval $]a, b[$ and linearly independent on this interval at $f(t) = 0$, and $\tilde{x}(t)$ is any partial solution of Eq. (1.1), defined on $]a, b[$.

To select a certain solution from the general solution (1.2) of Eq. (1.1), it is necessary to specify some additional conditions. In Sec. "Differential Equations" of the course of calculus, the Cauchy problem was solved. At some point $t_0 \in]a, b[$, values of the unknown function and its first derivative were specified:

$$x(t_0) = x_0, \qquad x'(t_0) = x'_0. \tag{1.3}$$

With that, it was noted that there exists one and only one solution of the problem (1.1), (1.3). However, in some particular physical problems it is often required to select from the set of solutions (1.2) such ones which satisfy at the ends of the interval $[a, b]$ the following conditions:

$$\begin{aligned} \alpha_1x(a) + \alpha_2x'(a) &= x_a, \\ \beta_1x(b) + \beta_2x'(b) &= x_b, \end{aligned} \tag{1.4}$$

where α_1, α_2, β_1, β_2, x_a, and x_b are constant numbers such that at least one of them in the pairs α_1 and α_2, β_1 and β_2 would be other than zero (i.e., $|\alpha_1| + |\alpha_2| \neq 0$ and $|\beta_1| + |\beta_2| \neq 0$). In these problems, values of the sought-for function are specified at two end points of the interval on which the solution is to be found.

An example may be the problem on a material point of mass m moving under the action of a given force $F(t, x, x')$. In such a problem, the law of motion is often to be found if at the initial time $t = t_0$ the point was at the position x_0 and at the time $t = t_1$ it should arrive at the point $x = x_1$. The solution of the problem is reduced to the integration of the Newton equation

$$m\frac{d^2x}{dt^2} = F(t, x, x')$$

with the boundary conditions $x(t_0) = x_0$, $x(t_1) = x_1$.

Note that this problem, generally speaking, does not have a unique solution. If we deal with a ballistic problem, a body can arrive at one and the same point when moving along a plunging or a flat trajectory and, moreover, if the initial velocity is very high, having circled around the earth one or several times.

◊ If the general solution of a differential equation is known, to solve the boundary value problem, it is necessary to find the arbitrary constants involved in the general solution from the boundary conditions. In this case, a real solution does not always exist, and, if it does, it is not necessarily unique.

Example 1.1. Solve the differential equation

$$y'' + y = 0 \qquad \text{for} \qquad y(0) = 0, \quad y(t_1) = y_1.$$

Solution. The original equation is a second order linear ordinary differential equation with constant coefficients. The general solution of this equation has the form

$$y = C_1 \cos t + C_2 \sin t.$$

From the first boundary condition it follows that $C_1 = 0$. Then $y(t) = C_2 \sin t$. If $t_1 \neq \pi n$, then from the second boundary condition we find

$$y_1 = y(t_1) = C_2 \sin t_1, \qquad \text{i.e.,} \qquad C_2 = \frac{y_1}{\sin t_1}.$$

Hence, in this case there exists a unique solution of the boundary value problem:

$$y(t) = \frac{y_1}{\sin t_1} \sin t.$$

If $t_1 = \pi n$ and $y_1 = 0$, then all the curves of the bundle $y = C_2 \sin t$ are solutions of the boundary value problem.

For $t_1 = \pi n$ and $y_1 \neq 0$ this problem has no solution.

In applications to the problems of mathematical physics we shall be interested in the main in homogeneous boundary value problems, that is, in finding solutions to homogeneous linear equations with homogeneous boundary conditions.

♦ Boundary conditions are called homogeneous if from the fact that functions $x_1(t), \ldots, x_n(t)$ satisfy these conditions it follows that any linear combination of these functions $x(t) = \sum\limits_{k=1}^{n} C_k x_k(t)$ also satisfies these conditions.

◊ Conditions (1.4) are homogeneous if $x_a = x_b = 0$.

◊ In what follows we shall not be interested in the case of trivial (i.e., identically equal to zero) solutions.

⧫ An equation of the form

$$\frac{d}{dt}[\varphi(t)x'(t)] - q(t)x(t) = f(t) \tag{1.5}$$

is called a selfadjoint differential equation. The functions $f(t)$ and $q(t)$ are assumed to be continuous on the interval $[a, b]$ and $\varphi(t)$ to be continuous together with its derivative.

Statement 1.1. *Equation* (1.1) *can be written in selfadjoint form.*

Take the function

$$\varphi(t) = \exp\left[\int_0^t p(\tau)d\tau\right]$$

and note that $\varphi'(t) = p(t)\varphi(t)$; hence,

$$\varphi(t)x''(t) + p(t)\varphi(t)x'(t) = \frac{d}{dt}[\varphi(t)x'(t)].$$

Multiplying (1.1) by $\varphi(t)$, we obtain

$$\frac{d}{dt}[\varphi(t)x'(t)] - \tilde{q}(t)x(t) = \tilde{f}(t), \tag{1.6}$$

where $\tilde{q}(t) = -\varphi(t)q(t)$ and $\tilde{f}(t) = f(t)\varphi(t)$. Thus, we have reduced Eq. (1.1) to a selfadjoint equation.

Example 1.2. Write the equation

$$x'' + \frac{1}{t}x' + \left(1 - \frac{\nu^2}{t^2}\right)x = 0, \qquad x = x(t)$$

in selfadjoint form.

Solution. Multiply the left and the right sides of the equation by the function

$$\varphi(t) = \exp\left(\int_1^t \frac{d\tau}{\tau}\right) = t,$$

to get

$$tx'' + x' + \left(t - \frac{\nu^2}{t}\right)x = 0$$

or

$$\frac{d}{dt}[tx'] + \left(t - \frac{\nu^2}{t}\right)x = 0.$$

Example 1.3. Solve the boundary value problem

$$x'' - x = 2t, \quad x = x(t), \qquad x(0) = 0, \quad x(1) = -1. \tag{1.7}$$

Solution. 1. Find the general solution to the homogeneous equation

$$x'' - x = 0.$$

Write the characteristic equation

$$k^2 - 1 = 0,$$

Hence, $k = \pm 1$, and the general solution of the homogeneous equation has the form

$$\bar{x}(t) = \widetilde{C}_1 e^t + \widetilde{C}_2 e^{-t} = C_1 \operatorname{ch} t + C_2 \operatorname{sh} t.$$

A particular solution of the nonhomogeneous equation is the function

$$\tilde{x}(t) = -2t.$$

Thus, for the general solution of Eq. (1.7) we obtain

$$x(t) = C_1 \operatorname{ch} t + C_2 \operatorname{sh} t - 2t. \tag{1.8}$$

2. Choose the constants C_1 and C_2 from the boundary conditions. Substituting (1.8) into (1.7):

$$\begin{cases} C_1 = 0, \\ C_1 \operatorname{ch} 1 + C_2 \operatorname{sh} 1 - 2 = -1, \end{cases}$$

we obtain

$$C_1 = 0, \qquad C_2 = \frac{1}{\operatorname{sh} 1}.$$

Finally,

$$x(t) = \frac{\operatorname{sh} t}{\operatorname{sh} 1} - 2t.$$

2 The Sturm–Liouville Problem

An important case of homogeneous boundary value problems are the so-called eigenvalue problems. A typical eigenvalue problem for a linear differential equation is the Sturm–Liouville problem. Consider the linear second order ordinary differential equation

$$\frac{d}{dt}[\varphi(t)x'(t)] - q(t)x(t) + \lambda\rho(t)x(t) = 0 \tag{2.1}$$

with the boundary conditions

$$\begin{aligned} \alpha_1 x'(a) + \alpha_2 x(a) = 0, &\qquad \beta_1 x'(b) + \beta_2 x(b) = 0; \\ |\alpha_1| + |\alpha_2| \neq 0, &\qquad |\beta_1| + |\beta_2| \neq 0. \end{aligned} \tag{2.2}$$

Assume that the functions $\varphi'(t)$, $q(t)$, and $\rho(t)$ are continuous on an interval $[a, b]$. Further we believe that $\varphi(t) > 0$, $\rho(t) > 0$, and $q(t) \geqslant 0$. The number λ is the parameter of the equation and α_1, α_2, β_1, and β_2 are preassigned constants.

♦ The problem of finding the values of the parameter λ at which there exist nontrivial solutions $x_\lambda(t)$ of Eq. (2.1) which satisfy the boundary conditions (2.2) is called the Sturm–Liouville problem. The values of the parameter λ at which there exist solutions of the Sturm–Liouville problem (2.1) and (2.2) are called the eigenvalues and the corresponding solutions $x_\lambda(t)$ are called the eigenfunctions of the problem. The boundary conditions (2.2) are called the Sturm boundary conditions.

♦ The set of all eigenvalues of the Sturm–Liouville problem is called a spectrum.

The eigenfunctions of the Sturm–Liouville problem have a number of remarkable properties which are widely utilized in solving boundary value problems not only for ordinary differential equations, but also for partial differential equations.

Properties of the eigenvalues and eigenfunctions of the Sturm–Liouville problem

Property 1. *There exists a sequence of eigenvalues* $\{\lambda_n\}$, $n = \overline{1, \infty}$, *and the corresponding sequence of eigenfunctions* $\{x_n(t)\}$, $n = \overline{1, \infty}$, *of the Sturm–Liouville problem (2.1), (2.2), and all eigenvalues can be ordered so that*

$$|\lambda_1| \leqslant |\lambda_2| \leqslant \ldots \leqslant |\lambda_n| \leqslant \ldots \tag{2.3}$$

Proof will be given below in Sec. "Characteristic numbers and eigenfunctions" of Ch. "Integral Equations" of [2].

Property 2. *To every eigenvalue corresponds, up to a constant multiplier, only one eigenfunction.*

Proof. Assume that an eigenvalue λ is associated with two eigenfunctions $x(t)$ and $y(t)$. Then from (2.1) it follows that

$$\frac{d}{dt}[\varphi(t)x'(t)] - q(t)x(t) + \lambda\rho(t)x(t) = 0,$$
$$\frac{d}{dt}[\varphi(t)y'(t)] - q(t)y(t) + \lambda\rho(t)y(t) = 0.$$

Multiply the first equation by $y(t)$ and the second one by $x(t)$ and subtract the latter from the former to get

$$y(t)\frac{d}{dt}[\varphi(t)x'(t)] - x(t)\frac{d}{dt}[\varphi(t)y'(t)] +$$
$$+x'(t)\varphi(t)y'(t) - x'(t)\varphi(t)y'(t) = 0.$$

The last equality can be written

$$\frac{d}{dt}[y(t)\varphi(t)x'(t) - x(t)\varphi(t)y'(t)] = 0,$$

which yields

$$\varphi(t)[y(t)x'(t) - x(t)y'(t)] = \text{const}, \quad t \in [a, b]. \tag{2.4}$$

From the boundary conditions (2.2) it follows that

$$\alpha_1 x(a) + \alpha_2 x'(a) = 0,$$
$$\alpha_1 y(a) + \alpha_2 y'(a) = 0,$$

and, by condition (2.2), we have $|\alpha_1| + |\alpha_2| \neq 0$. Hence,

$$\begin{vmatrix} x(a) & x'(a) \\ y(a) & y'(a) \end{vmatrix} = x(a)y'(a) - y(a)x'(a) = 0,$$

and, since the right side of (2.4) is independent of t and the left side vanishes at the point a, then the constant in the right side is equal to zero. As a result, the Wronskian is

$$W[x(t), y(t)] = y(t)x'(t) - x(t)y'(t) = \begin{vmatrix} y(t) & x(t) \\ y'(t) & x'(t) \end{vmatrix} = 0$$

for $t \in [a, b]$, that is, the functions $x(t)$ and $y(t)$ are linearly dependent and, hence, $x(t) = Cy(t)$, Q. E. D.

♦ The functions $x(t)$ and $y(t)$, defined and integrable on an interval $]a,b[$, are called orthogonal on this interval with weight function $\rho(t)$ if

$$\langle x(t)|y(t)\rangle_\rho = \int\limits_a^b x(t)y(t)\rho(t)dt = 0, \tag{2.5}$$

where $\rho(t) > 0$ is defined and integrable on $]a,b[$.

♦ The number assigned to each pair of real functions $x(t)$ and $y(t)$ by the rule (2.5) is called by a scalar product of the functions $x(t)$ and $y(t)$ with weight function $\rho(t) > 0$ on $]a,b[$.

♦ The quantity

$$\|x\| = \|x(t)\| = \sqrt{\langle x(t)|x(t)\rangle_\rho} \tag{2.6}$$

is called the norm of the function $x(t)$.

Property 3. *The eigenfunctions of the Sturm–Liouville problem corresponding to different eigenvalues are pairwise orthogonal on the interval $]a,b[$ with weight function $\rho(t)$.*

Proof. Let $x_k(t)$ and $x_l(t)$ be the eigenfunctions corresponding to eigenvalues λ_k and λ_l $(k \neq l)$. Then

$$\frac{d}{dt}[\varphi(t)x'_k(t)] - q(t)x_k(t) = -\lambda_k\rho(t)x_k(t),$$
$$\frac{d}{dt}[\varphi(t)x'_l(t)] - q(t)x_l(t) = -\lambda_l\rho(t)x_l(t).$$

Multiplying the first equation by $x_l(t)$ and the second one by $x_k(t)$ and subtracting the latter from the former, we obtain

$$x_l(t)\frac{d}{dt}[\varphi(t)x'_k(t)] - x_k(t)\frac{d}{dt}[\varphi(t)x'_l(t)] + \varphi(t)x'_l(t)x'_k(t) - \varphi(t)x'_l(t)x'_k(t) = \\ = (\lambda_l - \lambda_k)\rho(t)x_k(t)x_l(t)$$

or

$$\frac{d}{dt}\{\varphi(t)[x_l(t)x'_k(t) - x_k(t)x'_l(t)]\} = (\lambda_l - \lambda_k)\rho(t)x_k(t)x_l(t).$$

Integrate the last equality with respect to t from a to b and take into account the relations

$$x_l(a)x'_k(a) - x_k(a)x'_l(a) = x_l(b)x'_k(b) - x_k(b)x'_l(b) = 0,$$

which are true due to the condition that there exist nontrivial solutions of the following systems of algebraic equations:

$$\begin{cases} \alpha_1 x_k(a) + \alpha_2 x'_k(a) = 0, \\ \alpha_1 x_l(a) + \alpha_2 x'_l(a) = 0, \end{cases} \quad \text{and} \quad \begin{cases} \beta_1 x_k(b) + \beta_2 x'_k(b) = 0, \\ \beta_1 x_l(b) + \beta_2 x'_l(b) = 0. \end{cases}$$

As a result, we obtain

$$(\lambda_l - \lambda_k)\int\limits_a^b \rho(t)x_k(t)x_l(t)dt = 0.$$

Since $\lambda_l \neq \lambda_k$, this proves Property 3.

◊ From Property 2 it follows that the eigenfunctions are determined up to a constant. This constant is often found from the condition that the norm of the function $x_k(t)$ is equal to unity:

$$\|x_k(t)\|^2 = \int_a^b \rho(t)x_k^2(t)dt = 1, \qquad k = \overline{1,\infty}. \tag{2.7}$$

Equality (2.7) is called the normalization condition.

♦ A set of functions $\{x_k(t)\}$, $k = \overline{1,\infty}$, is called orthogonal on an interval $]a, b[$ with weight function $\rho(t)$ if, for any $k, l = \overline{1,\infty}$, the relation $\langle x_k(t)|x_l(t)\rangle_\rho = 0$, $k \neq l$ is valid.

♦ A set of functions $\{x_k(t)\}$, $k = \overline{1,\infty}$, which are orthogonal on an interval $]a, b[$, is called orthonormal with weight function $\rho(t)$ if $\|x_k(t)\| = 1$, $k = \overline{1,\infty}$.

The set of functions $\{x_k(t)\}$, $k = \overline{1,\infty}$, which are orthogonal on $]a, b[$ generates a set of functions $\{u_n(t)\}$, which are orthonormal with weight function $\rho(t) = 1$. Here,

$$u_n(t) = \frac{\sqrt{\rho(t)}x_n(t)}{\|x_k(t)\|}. \tag{2.8}$$

◊ As a result, the completeness condition (see Sec. "The delta function and orthonormal systems" of [2]) for the orthogonal set of functions with weight function $\rho(t)$ will take the form

$$\sum_{k=1}^{\infty} \frac{x_k(t)x_k(\tau)}{\|x_k(t)\|^2}\rho(\tau) = \delta(t-\tau), \tag{2.9}$$

where $\delta(t-\tau)$ is the Dirac delta function.

Property 4 (Steklov's expansion theorem). *If a function $f(t)$ is twice continuously differentiable on $[a, b]$ and satisfies the boundary conditions (2.2), it can be expanded in a series in the eigenfunctions $x_k(t)$ of the Sturm–Liouville problem (2.1), (2.2) absolutely and uniformly converging on $[a, b]$*

$$f(t) = \sum_{k=1}^{\infty} C_k x_k(t), \tag{2.10}$$

where

$$C_k = \frac{\langle f(t)|x_k(t)\rangle_\rho}{\|x_k(t)\|^2} = \frac{\int_a^b f(t)x_k(t)\rho(t)dt}{\int_a^b x_k^2(t)\rho(t)dt}. \tag{2.11}$$

The series (2.10) is called the Fourier series of the function $f(t)$ in the set of orthogonal functions $\{x_k(t)\}$, and the coefficients (2.11) are called the Fourier coefficients of the function $f(t)$.

Proof was given in Sec. "Hilbert–Schmidt theorem and its corollaries" of Chap. "Integral Equations" of [3]).

Example 2.1. Show that the set of eigenfunctions of the Sturm–Liouville problem (2.1), (2.2) satisfies the completeness condition (2.9) on an interval $[a, b]$ in the class of functions twice continuously differentiable on this interval if these functions satisfy the boundary conditions (2.2).

Solution. Consider the series (2.10) where the function $f(t)$ satisfies the conditions of Steklov's theorem. In view of (2.11), we obtain

$$f(t) = \sum_{k=1}^{\infty} C_k x_k(t) = \sum_{k=1}^{\infty} x_k(t) \int_a^b \frac{f(\tau)x_k(\tau)\rho(\tau)}{\|x_k\|^2} d\tau.$$

Interchange the summation and the integration. This can be done since the series (2.10) converges uniformly. Hence,

$$f(t) = \int_a^b f(\tau) \left\{ \sum_{k=1}^{\infty} \frac{x_k(\tau)x_k(t)}{\|x_k\|^2} \rho(\tau) \right\} d\tau = \int_a^b f(\tau)\delta(t-\tau)d\tau.$$

Here, we have made use of the definition of the Dirac delta function. Thus, the completeness condition is fulfilled.

Example 2.2. Find the eigenvalues and orthonormal eigenfunctions of the Sturm–Liouville problem

$$y'' + \lambda y = 0, \quad y = y(x), \qquad y(0) = y(l) = 0, \qquad l > 0.$$

and write the orthogonality relationship for the eigenfunctions.

Solution. 1. Let $\lambda = 0$. Then

$$y(x) = C_1 x + C_2,$$

and the boundary conditions are satisfied only for the trivial solution $C_1 = C_2 = 0$.

2. Let $\lambda < 0$. Then the general solution of the equation $y'' + \lambda y = 0$ is the function

$$y(x) = C_1 e^{x\sqrt{-\lambda}} + C_2 e^{-x\sqrt{-\lambda}}.$$

From the boundary conditions we find $C_1 = C_2 = 0$, that is there exists only the trivial solution.

3. Let $\lambda > 0$. Then the general solution of the equation $y'' + \lambda y = 0$ has the form

$$y(x) = C_1 \sin x\sqrt{\lambda} + C_2 \cos x\sqrt{\lambda}.$$

Substituting this expression into the boundary conditions, we obtain from the first one $C_2 = 0$ and from the second one

$$C_1 \sin(\sqrt{\lambda} l) = 0.$$

Thus, this boundary value problem has a nontrivial solution if

$$l\sqrt{\lambda} = \pi n \quad \text{and} \quad \lambda_n = \left(\frac{\pi n}{l}\right)^2, \quad n = \overline{1, \infty}.$$

To this values correspond the eigenfunctions

$$y_n(x) = C_n \sin \frac{\pi n x}{l}, \qquad n = \overline{1, \infty}.$$

4. Write the orthogonality relationship. To this end, calculate the integral

$$I = \int_0^l y_k(x) y_n(x) \rho(x) dx.$$

In our case, $\rho(x) = 1$ since the original selfadjoint equation (2.1) has the form $y'' - \lambda y = 0$. Then

$$I = \int_0^l C_k \sin\frac{\pi n x}{l} C_n \sin\frac{\pi n x}{l} dx = \delta_{kn} C_k^2 \int_0^l \sin^2\frac{\pi n x}{l} dt = \delta_{kn} C_k^2 \frac{l}{2}.$$

Finally,

$$I = \int_0^l y_k(x) y_n(x) dx = \frac{lC_k^2}{2}\delta_{kn}.$$

If we put $C_k = \sqrt{2/l}$, the set of functions $y_n(x)$ will be orthonormal.

5. Consider a possible physical interpretation of the problem. Let, for example, a homogeneous elastic rod aligned with the Ox-axis is compressed along the axis by a force P, and the rod ends located at the points $x = 0$ and $x = l$ are fixed at the axis, but may freely revolve around the fixation points (Fig. 1).

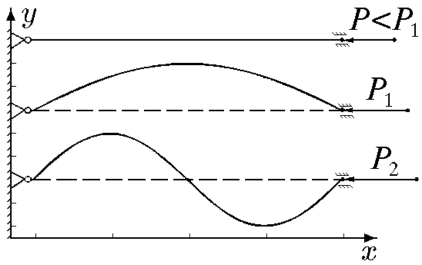

Fig. 1.

If we denote by y the transverse deviation of a point of the rod from its initial (rectilinear) position, it appears that the function $y(x)$ satisfies, with a sufficient accuracy, the differential equation

$$y''(x) + \frac{P}{EJ} y(x) = 0$$

and the boundary conditions

$$y(0) = y(l) = 0,$$

where E, Young's modulus, and J, the so-called "moment of inertia", characterize the rod material and its cross section [9]. It is natural that for small values of P the rectilinear shape of the rod is stable. However, there exists a critical value P_1 of the force P, such that for $P > P_1$ the rectilinear shape of the rod becomes unstable and the rod is bounded. The critical values P_n at which the rod takes a stable curvilinear shape are given by the expression for the eigenvalues λ_n

$$\frac{P_n}{EJ} = -\lambda_n = \left(\frac{\pi n}{l}\right)^2$$

or

$$P_n = EJ\left(\frac{\pi n}{l}\right)^2, \qquad n = \overline{1, \infty}.$$

In this case, the eigenvalues determine, up to a multiplier (amplitude), the corresponding equilibrium states of the rod (see Fig. 1).

It can easily be seen that as the force direction (its sign in the equation) is reversed (tension), it appears that the equilibrium state is represented only by the straight line $y(x) = 0$, completely according to the sign of the eigenvalues λ_n.

The simplest solution for $n = 1$ was found by Euler as early as 1757. It should however be noted that the equation $y'' = \lambda y$ describes small deviations, while analysis of a more exact equation which holds for any deviations (it appears to be linear) shows that as P_n is increased, the maximum deviation of segments of a rod being in the equilibrium state from the straight line $y(x) = 0$ rapidly increases, and the rod is destroyed.

The above physical interpretation of the mathematical solution is not unique. Similar problems arise in finding natural oscillations of strings, rods, etc.

◊ Eigenvalue problems arise in the main when solving mixed problems for partial differential equations. If such a mixed problem admits separation of variables, it can be reduced to the "spectral" eigenvalue problem (2.1) (see Secs. "Separation of variables in Laplace's equation", "Separation of variables in Helmholtz's equation", "The Fourier method for the heat equation", "The Fourier method for the wave equation" of [3]).

In conclusion we consider some examples which illustrate the influence of the boundary conditions and the equation coefficients on the solution of the Sturm–Liouville problem.

Example 2.3. Solve the Sturm–Liouville problem

$$x'' - \lambda x = 0, \qquad x = x(t), \qquad x'(1) = x'(3) = 0, \tag{2.12}$$

write the orthogonality relationship for the eigenfunctions and orthonormalize these functions.

Solution. 1. Find the general solution to equation (2.12). Let us construct a characteristic equation:

$$k^2 - \lambda = 0.$$

Hence, we have $k = \pm\sqrt{\lambda}$ and, depending on the character of the values of λ, obtain the following general solutions:

$$\begin{array}{lll} \text{(a)} & \lambda > 0 & x(t) = \widetilde{C}_1 e^{\sqrt{\lambda}t} + \widetilde{C}_2 e^{-\sqrt{\lambda}t} = C_1 \operatorname{ch}\sqrt{\lambda}t + C_2 \operatorname{sh}\sqrt{\lambda}t; \\ \text{(b)} & \lambda = 0 & x(t) = C_1 + C_2 t; \\ \text{(c)} & \lambda < 0 & x(t) = C_1 \cos\sqrt{-\lambda}t + C_2 \sin\sqrt{-\lambda}t. \end{array}$$

2. Consider the case $\lambda > 0$. Then

$$x'(t) = \sqrt{\lambda}(C_1 \operatorname{sh}\sqrt{\lambda}t + C_2 \operatorname{ch}\sqrt{\lambda}t),$$

and to determine the constants C_1, C_2 and λ we obtain from the boundary conditions

$$\begin{cases} \sqrt{\lambda}(C_1 \operatorname{sh}\sqrt{\lambda} + C_2 \operatorname{ch}\sqrt{\lambda}) = 0, \\ \sqrt{\lambda}(C_1 \operatorname{sh} 3\sqrt{\lambda} + C_2 \operatorname{ch} 3\sqrt{\lambda}) = 0. \end{cases}$$

From the first equation we find

$$C_1 = -C_2 \frac{\operatorname{ch}\sqrt{\lambda}}{\operatorname{sh}\sqrt{\lambda}}.$$

Substituting this expression into the second equation, we get

$$\frac{C_2}{\operatorname{sh}\sqrt{\lambda}}\big(- \operatorname{ch}\sqrt{\lambda} \operatorname{sh} 3\sqrt{\lambda} + \operatorname{sh}\sqrt{\lambda} \operatorname{ch} 3\sqrt{\lambda}\big) = 0.$$

In view of $\operatorname{ch} a \operatorname{sh} b \pm \operatorname{sh} a \operatorname{ch} b = \operatorname{sh}(a \pm b)$, we find

$$C_2 \operatorname{sh}(2\sqrt{\lambda}) = 0.$$

Hence, $C_2 = 0$ or $\sqrt{\lambda} = 0$. The latter is impossible since $\lambda > 0$. Thus, for $\lambda > 0$ we have $C_1 = C_2 = 0$, and there are no nontrivial solutions.

3. Consider the case $\lambda = 0$. From the boundary conditions we find $C_2 = 0$. The constant C_1 cannot be determined from the boundary conditions. For convenience we redenote $C_1 = C_0$. Hence for $\lambda = 0$ we have

$$x(t) = C_0. \tag{2.13}$$

4. Consider the case $\lambda < 0$. Similar to case 2, we obtain

$$x'(t) = \sqrt{-\lambda}(-C_1 \sin\sqrt{-\lambda}t + C_2 \cos\sqrt{-\lambda}t)$$

and

$$\begin{cases} \sqrt{-\lambda}(-C_1 \sin\sqrt{-\lambda} + C_2 \cos\sqrt{-\lambda}) = 0, \\ \sqrt{-\lambda}(C_1 \sin 3\sqrt{-\lambda} + C_2 \cos 3\sqrt{-\lambda}) = 0. \end{cases}$$

Multiply the first equation by $\sin 3\sqrt{-\lambda}$ and the second one by $-\sin\sqrt{-\lambda}$ and then combine the resulting expressions to get

$$\sqrt{-\lambda}(\cos\sqrt{-\lambda}\sin 3\sqrt{-\lambda} - \sin\sqrt{-\lambda}\cos 3\sqrt{-\lambda})C_2 = 0$$

or

$$\sqrt{-\lambda}C_2 \sin 2\sqrt{-\lambda} = 0.$$

Nontrivial solutions of the Sturm–Liouville problem exist if

$$2\sqrt{-\lambda} = \pi k \quad \text{and} \quad -C_1 \sin\frac{\pi k}{2} + C_2 \cos\frac{\pi k}{2} = 0, \quad k = \overline{1, \infty}.$$

Denoting

$$C_1 = C_k \cos\frac{\pi k}{2}, \qquad C_2 = C_k \sin\frac{\pi k}{2},$$

we finally obtain

$$\lambda_k = -\Big(\frac{\pi k}{2}\Big)^2,$$

$$x_k(t) = C_k\Big(\cos\frac{\pi k}{2}\cos\frac{\pi k t}{2} + \sin\frac{\pi k}{2}\sin\frac{\pi k t}{2}\Big) = C_k \cos\frac{\pi k}{2}(t-1), \quad k = \overline{1, \infty}. \tag{2.14}$$

5. Combining (2.13) and (2.14) yields

$$\lambda_k = -\Big(\frac{\pi k}{2}\Big)^2, \quad x_k(t) = C_k(1 + \delta_{k0})\cos\frac{\pi k}{2}(t-1), \quad k = \overline{0, \infty}. \tag{2.15}$$

6. Write the orthogonality relationship. Equation (2.12) is written in the selfadjoint form with $\rho(t) = 1$. Hence, the functions (2.15) should be orthogonal with respect to the scalar product

$$\langle x_k(t), x_l(t)\rangle = \int_1^3 x_k(t)x_l(t)dt.$$

In view of the integrals

$$\int_1^3 \cos\frac{\pi k}{2}(t-1)dt = 0, \qquad \int_1^3 dt = 2,$$

$$\int_1^3 \cos\frac{\pi k}{2}(t-1)\cos\frac{\pi l}{2}(t-1)dt = \delta_{kl}, \quad k, l = \overline{1, \infty},$$

we obtain the sought-for orthogonality relationship for the eigenfunctions (2.15)

$$\int_1^3 x_k(t)x_l(t)dt = C_k^2(1+\delta_{k0})\delta_{kl}, \qquad k,l=\overline{0,\infty}.$$

7. The orthonormal eigenfunctions are obtained from (2.15) by dividing it by the norm of the eigenfunction (square root of the coefficient at δ_{kl} in the last relation):

$$x_k(t) = \frac{1}{\sqrt{1+\delta_{k0}}}\cos\frac{\pi k}{2}(t-1).$$

♢ All statements in this section have been made under the assumption that $\varphi(t) > 0$ for $t \in [a,b]$. They however remain valid if $\varphi(a) = 0$ and/or $\varphi(b) = 0$, and $\varphi(t)$ can be represented as $\varphi(t) = (t-a)\psi(t)$ and $\psi(a) \neq 0$ and/or $\varphi(t) = (t-b)\psi(t)$ and $\psi(b) \neq 0$. Then it is necessary to impose, at the corresponding boundary point, the condition

$$\Big|\lim_{t\to a+0} x(t)\Big| < \infty \qquad \text{and(or)} \qquad \Big|\lim_{t\to b-0} x(t)\Big| < \infty. \tag{2.16}$$

The condition (2.16) can be replaced by the condition that the function $x(t)$ is square integrable on the interval $]a,b[$. These statements will also remain valid for unbounded intervals, i.e., $a = -\infty$ and/or $b = \infty$. Problems for which all above statements hold true are called singular. For more details see the Sturm–Liouville problem for the equations of Bessel, Legendre, Laguerre, etc.

Example 2.4. Solve the Sturm–Liouville problem

$$y'' + \frac{2}{t}y' + \lambda y = 0, \quad y = y(t), \qquad 0 < t < l, \tag{2.17}$$
$$|\lim_{t\to 0} y(t)| < \infty, \qquad y(l) = 0.$$

Write the orthogonality relationship for the eigenfunctions of the problem and orthonormalize these functions.

Solution. Make in Eq. (2.17) the change

$$y(t) = \frac{z(t)}{t}. \tag{2.18}$$

Then

$$y' = \frac{z'}{t} - \frac{z}{t^2}, \qquad y'' = \frac{z''}{t} - 2\frac{z'}{t^2} + 2\frac{z}{t^3}.$$

Substituting these expressions into (2.17), we obtain

$$\Big[\frac{z''}{t} - 2\frac{z'}{t^2} + 2\frac{z}{t^3}\Big] + \frac{2}{t}\Big[\frac{z'}{t} - \frac{z}{t^2}\Big] + \lambda\frac{z}{t} = 0;$$
$$\Big|\lim_{t\to 0}\frac{z(t)}{t}\Big| < \infty, \qquad \frac{z(l)}{l} = 0.$$

As a result, for the determination of the function $z = z(t)$, we have the following Sturm–Liouville problem:

$$z'' + \lambda z = 0, \qquad z(0) = z(l) = 0, \quad 0 < t < l.$$

The solution of this problem, obtained in Example 2.2, is

$$\lambda_n = \left(\frac{\pi n}{l}\right)^2, \qquad z_n(t) = C_n \sin\frac{\pi n}{l}t, \quad n = \overline{1,\infty} \tag{2.19}$$

with the orthogonality relationship

$$\int\limits_0^l z_n(t)z_k(t)dt = C_n^2\frac{l}{2}\delta_{kn}, \qquad k,n = \overline{1,\infty}. \tag{2.20}$$

From (2.19) and (2.20), in view of (2.18), we obtain the following solution of the original problem:

$$\lambda_n = \left(\frac{\pi n}{l}\right)^2, \qquad y_n(t) = \frac{C_n}{t}\sin\frac{\pi n}{l}t,$$

and the orthogonality relationship can be written as

$$\int\limits_0^l t^2 y_n(t)y_k(t)dt = C_n^2\frac{l}{2}\delta_{nk}, \quad n,k = \overline{1,\infty}.$$

The orthonormal eigenfunctions will take the form

$$y_n(t) = \sqrt{\frac{2}{l}}\frac{1}{t}\sin\frac{\pi n}{l}t.$$

The mathematical statement of the majority of physical problems involves Sturm's boundary conditions (2.2). However, there exists an important class of boundary conditions – the so-called periodic boundary conditions:

$$y(t+l) = y(t), \quad t \in \mathbb{R}, \quad l > 0,$$

or, in equivalent form,

$$y(a) - y(b) = 0, \quad y'(a) - y'(b) = 0, \quad a \in \mathbb{R}, \quad b = a + l.$$

The principal difference of the periodic boundary conditions (the term is clear from their explicit form) from Sturm's boundary conditions (2.2) is that for second order equations, two linearly independent eigenfunctions may correspond to one eigenvalue λ_n.

Example 2.5. Find the eigenvalues and eigenfunctions of the problem

$$y'' + \lambda y = 0, \qquad y(0) = y(l), \quad y'(0) = y'(l).$$

Solution. The general solution of the original equation has the form

$$y(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}.$$

Substitution of this equation into the boundary conditions yields the system of equations

$$C_1(e^{\sqrt{-\lambda}l} - 1) + C_2(e^{-\sqrt{-\lambda}l} - 1) = 0,$$
$$C_1(e^{\sqrt{-\lambda}l} - 1) - C_2(e^{-\sqrt{-\lambda}l} - 1) = 0.$$

This system of equations will have nontrivial solutions if its determinant is equal to zero. In this case, we arrive at the equation

$$(e^{\sqrt{-\lambda}l} - 1)(e^{-\sqrt{-\lambda}l} - 1) = 0,$$

from which

$$\sqrt{-\lambda}l = 2n\pi i, \qquad n = \overline{0, \infty},$$

and, hence,

$$\lambda_n = \Big(\frac{2n\pi}{l}\Big)^2.$$

Thus, two eigenfunctions, $\sin(2\pi nx/l)$ and $\cos(2\pi nx/l)$, correspond to each of the eigenvalues λ_n with $n > 0$ and only one eigenfunction, $\cos 0 = 1$, corresponds to the eigenvalue $\lambda_0 = 0$.

$\Diamond$ The other properties of a problem with periodic boundary conditions coincide with those properties of the problem with Sturm's boundary conditions (2.2).

CHAPTER 2

Cylindrical Functions

3 Bessel functions of the first kind

Let us consider the equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad \operatorname{Re}\nu \geqslant 0. \tag{3.1}$$

♦ Equation (3.1) is called Bessel's equation of index ν, and its solutions not equal identically to zero are called cylindrical functions.

◊ Functions of this type arise in solving partial differential equations containing the Laplacian operator in cylindrical coordinates by the method of separation of variables. Therefore, they are called cylindrical.

Theorem 3.1. *The function*

$$y(x) = \sum_{k=0}^{\infty}(-1)^k \frac{2^\nu\Gamma(\nu+1)C_0}{k!\Gamma(\nu+1+k)}\Big(\frac{x}{2}\Big)^{2k+\nu}, \qquad C_0 \neq 0, \tag{3.2}$$

is a particular solution of Eq. (3.1) *with the series* (3.2) *converging uniformly in a finite interval* $[0,a]$.

Proof. We seek a particular solution of Eq. (3.1) in the form of a generalized power series

$$y(x) = \sum_{k=0}^{\infty} C_k x^{k+\sigma}, \tag{3.3}$$

where σ is some number and C_k are some constants to be determined, such that $C_0 \not\equiv 0$. Then

$$\begin{aligned}
&xy'(x) = C_0\sigma x^\sigma + C_1(\sigma+1)x^{\sigma+1} + \sum_{k=2}^{\infty} C_k(k+\sigma)x^{k+\sigma},\\
&x^2y''(x) = C_0\sigma(\sigma-1)x^\sigma + C_1(\sigma+1)\sigma x^{\sigma+1} + \sum_{k=2}^{\infty} c_k(k+\sigma)(k+\sigma-1)x^{k+\sigma},\\
&x^2y(x) = \sum_{n=0}^{\infty} C_n x^{n+\sigma+2} = \sum_{k=2}^{\infty} C_{k-2}x^{k+\sigma}.
\end{aligned} \tag{3.4}$$

In the above equality, we have made the change $n+2=k$. Substitute (3.2) and (3.3) into Eq. (3.1) to get

$$[\sigma(\sigma-1) + \sigma - \nu^2]x^\sigma C_0 + [\sigma(\sigma+1) + (\sigma+1) - \nu^2]x^{\sigma+1}C_1 +$$
$$+x^\sigma \sum_{k=2}^{\infty}\{[(k+\sigma)(k+\sigma-1) + (k+\sigma) - \nu^2]C_k + C_{k-2}\}x^k = 0.$$

Equating the coefficients of the terms with identical powers of x, we obtain

$$\begin{array}{ll} C_0[\sigma^2 - \nu^2] = 0 & x^\sigma, \\ C_1[(\sigma+1)^2 - \nu^2] = 0, & x^{\sigma+1}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots, & \\ C_k[(k+\sigma)^2 - \nu^2] + C_{k-2} = 0, & x^{\sigma+k}, \quad k > 2. \end{array}$$

Since $C_0 \neq 0$, from the first equation we obtain $\sigma = \pm\nu$.

1. Let $\sigma = \nu$. Then from the second equality we find $C_1 = 0$ and

$$C_k = -\frac{C_{k-2}}{(k+\nu)^2 - \nu^2} = -\frac{C_{k-2}}{k(2\nu + k)}.$$

Hence, for odd k $(k = 2l + 1)$

$$C_{2l+1} = 0,$$

and for even k $(k = 2l)$

$$C_{2l} = -\frac{C_{2(l-1)}}{2^2(\nu + l)l},$$

that is,

$$C_2 = -\frac{C_0}{2^2(\nu+1)1};$$

$$C_4 = \frac{C_0}{2^4 2!(\nu+1)(\nu+2)};$$

$$\vdots$$

$$C_{2l} = (-1)^l \frac{C_0}{2^{2l} l!(\nu+1)(\nu+2)\cdots(\nu+l)}.$$

It can easily be verified (e.g., applying the d'Alembert criterion) that the series (3.3) converges uniformly in any interval $[0, a]$, and, hence, the function $y(x)$ (3.3) is a solution of Bessel's equation for any C_0.

In view of the relations

$$z\Gamma(z) = \Gamma(z+1), \quad l! = \Gamma(l+1),$$
$$(\nu+1)(\nu+2)\cdots(\nu+l) = \frac{\Gamma(\nu+1)}{\Gamma(l+\nu+1)},$$

we obtain

$$C_{2l} = \frac{(-1)^l \Gamma(\nu+1) C_0}{2^{2l}\Gamma(l+\nu+1)\Gamma(l+1)},$$

and a solution of (3.1) is

$$y(x) = \sum_{l=0}^{\infty} (-1)^l \frac{\Gamma(\nu+1) 2^\nu C_0}{l!\Gamma(l+\nu+1)} \Big(\frac{x}{2}\Big)^{\nu+2l}, \tag{3.5}$$

which is the required result.

◊ It is convenient to use for C_0 the number

$$C_0 = \frac{1}{2^\nu \Gamma(\nu+1)}.$$

♦ The function

$$J_\nu(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\Gamma(\nu + k + 1)} \left(\frac{x}{2}\right)^{2k+\nu} \tag{3.6}$$

is called *a Bessel function of the first kind* (see Fig. 2). Here, the complex number ν is the order of the Bessel function and x is an independent variable.

◊ Since Eq. (3.1) does not change on changing ν for $-\nu$, the function $J_{-\nu}(x)$ is a solution of Eq. (3.1) as well.

Theorem 3.2. *The relation*

$$J_{-n}(x) = (-1)^n J_n(x) \tag{3.7}$$

is valid.

Proof. Consider the function

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k + 1 - n)} \left(\frac{x}{2}\right)^{2k-n}.$$

Note that

$$\Gamma(k + 1 - n) = \begin{cases} \infty & 0 \leqslant k < n, \\ (k - n)! & k \geqslant n. \end{cases}$$

Then,

$$J_{-n}(x) = \sum_{k=n}^{\infty} (-1)^k \frac{1}{k!(k - n)!} \left(\frac{x}{2}\right)^{2k-n}.$$

Putting $k = l + n$, in view of (3.6), we get

$$J_{-n}(x) = \sum_{l=0}^{\infty} (-1)^{l+n} \frac{1}{l!(l + n)!} \left(\frac{x}{2}\right)^{2l+n} = (-1)^n J_n(x), \tag{3.8}$$

proving the theorem.

Corollary. The relation

$$J_{-n}(-x) = J_n(x) \tag{3.9}$$

is valid.

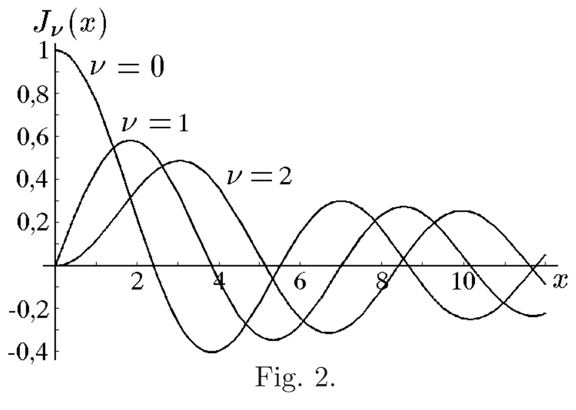

Fig. 2.

Proof. This relation follows from (3.8):

$$J_{-n}(-x) = \sum_{l=0}^{\infty} (-1)^{l+n} \frac{1}{l!(l+n)!} \left(\frac{-x}{2}\right)^{2l+n} = \sum_{l=0}^{\infty} (-1)^{3l+2n} \frac{1}{l!(l+n)!} \left(\frac{x}{2}\right)^{2l+n} = J_n(x).$$

Theorem 3.3. *If ν is not an integer, the functions $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent.*

Proof. Write out Bessel's equations for the functions J_ν and $J_{-\nu}$:

$$\frac{d}{dx}[xJ'_\nu(x)] + \left(x - \frac{\nu^2}{x}\right) J_\nu(x) = 0,$$
$$\frac{d}{dx}[xJ'_{-\nu}(x)] + \left(x - \frac{\nu^2}{x}\right) J_{-\nu}(x) = 0.$$

Multiply the first equation by $J_{-\nu}(x)$ and the second one by $J_\nu(x)$ and subtract the latter from the former to get

$$J_{-\nu}(x)\frac{d}{dx}[xJ'_\nu(x)] - J_\nu(x)\frac{d}{dx}[xJ'_{-\nu}(x)] + xJ'_{-\nu}(x)J'_\nu(x) - xJ'_\nu(x)J'_{-\nu}(x) =$$
$$= \frac{d}{dx}\{x[J_{-\nu}(x)J'_\nu(x) - J_\nu(x)J'_{-\nu}(x)]\} = -\frac{d}{dx}\{xW[J_\nu(x), J_{-\nu}(x)]\} = 0,$$

where $W[J_\nu, J_{-\nu}]$ is the Wronskian of the functions $J_\nu(x)$ and $J_{-\nu}(x)$. Hence,

$$xW[J_\nu(x), J_{-\nu}(x)] = C.$$

From here, in the limit of $x \to 0$, we can determine the constant C:

$$C = \lim_{x\to 0} xW[J_\nu(x), J_{-\nu}(x)] = \lim_{x\to 0}[xJ_\nu(x)J'_{-\nu}(x) - xJ_{-\nu}(x)J'_\nu(x)].$$

In view of the definition of a Bessel function (3.6) and the relation

$$xJ'_\nu(x) = \sum_{l=0}^{\infty} (-1)^l \frac{2l+\nu}{l!\Gamma(\nu+l+1)} \left(\frac{x}{2}\right)^{2l+\nu},$$

we find

$$C = \lim_{x\to 0}\left\{\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{k+l}(2l-\nu)}{k!\Gamma(\nu+k+1)l!\Gamma(-\nu+l+1)}\left(\frac{x}{2}\right)^{2k+2l} - \right.$$
$$\left. - \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{k+l}(2l+\nu)}{l!\Gamma(-\nu+k+1)k!\Gamma(\nu+l+1)}\left(\frac{x}{2}\right)^{2k+2l}\right\}.$$

Only the terms with $k = l = 0$ are nonzero, and therefore (see [2])

$$C = \frac{-\nu}{\Gamma(1+\nu)\Gamma(1-\nu)} - \frac{\nu}{\Gamma(1-\nu)\Gamma(1+\nu)} = -\frac{2\nu}{\nu\Gamma(\nu)\Gamma(1-\nu)} = \frac{-2}{\pi}\sin \pi\nu.$$

Here, we have made use of the relation

$$\Gamma(\nu)\Gamma(1-\nu) = \frac{\pi}{\sin \pi\nu}.$$

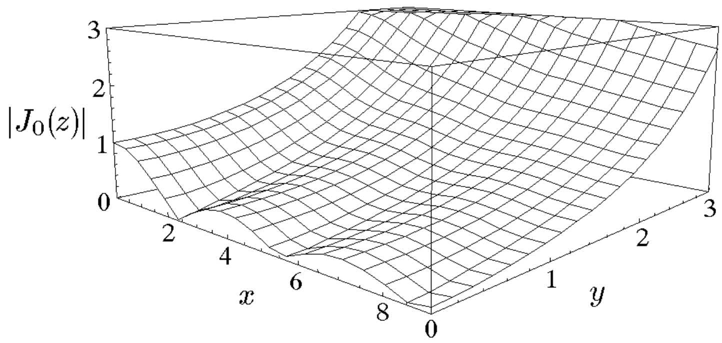

Fig. 3.

Thus,

$$C = -\frac{2}{\pi}\sin\pi\nu$$

and

$$W[J_\nu(x), J_{-\nu}(x)] = -\frac{2}{\pi x}\sin\pi\nu. \tag{3.10}$$

From (3.10) it follows, in particular, that $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent for noninteger ν, and thus the theorem is proved.

Corollary. If $\nu \neq n$, the general solution of Eq. (3.1) has the form

$$y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x). \tag{3.11}$$

Proof immediately follows from the fact that the functions $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent (see Eq. (3.10)) and are solutions of the second order linear differential equation (3.1).

$\Diamond$ Note that the Bessel functions (3.6) are defined not only for real, but also for complex arguments and orders. Thus, for example, Fig. 3 shows the modulus of a zero-order Bessel function of complex argument z and Fig. 4 presents plots of the real and imaginary parts and the modulus of the Bessel function $J_{1+i}(x)$ of complex order.

Example 3.1. Find the general solution of the equation

$$y'' + \frac{y'}{x} + \Big(1 - \frac{1}{9x^2}\Big)y = 0.$$

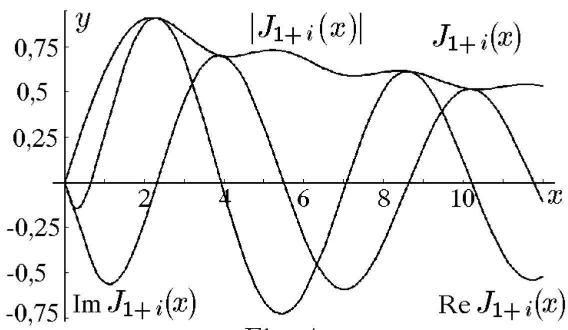

Fig. 4.

Solution. Write this equation as

$$x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0.$$

This is Bessel's equation with $\nu = 1/3$. Therefore, its general solution is

$$y(x) = C_1 J_{1/3}(x) + C_2 J_{-1/3}(x).$$

Example 3.2. Find the general solution of the equation

$$x^2y'' + xy' + 4(x^4 - 2)y = 0. \tag{3.12}$$

Solution. Make in Eq. (3.12) the change of variables $x^2 = \tau$, $x = \sqrt{\tau}$. Then,

$$y' = \frac{dy}{dx} = \frac{d\tau}{dx}\frac{dy}{d\tau} = 2x\frac{dx}{d\tau} = 2\sqrt{\tau}\frac{dy}{d\tau},$$
$$y'' = 2\sqrt{\tau}\frac{d}{d\tau}y' = \sqrt{\tau}\frac{d}{d\tau}\sqrt{\tau}\frac{dy}{d\tau} = 2\frac{dy}{d\tau} + 4\tau\frac{d^2y}{d\tau^2}.$$

Substituting these expressions into Eq. (3.12), we obtain

$$\tau\left(2\frac{dy}{d\tau} + 4\tau\frac{d^2y}{d\tau^2}\right) + 2\sqrt{\tau}\sqrt{\tau}\frac{dy}{d\tau} + 4(\tau^2 - 2)y = 0$$

or

$$\tau^2\frac{d^2y}{d\tau^2} + \tau\frac{dy}{d\tau} + (\tau^2 - 2)y = 0.$$

This is Bessel's equation of an order $\nu = \sqrt{2}$. Its general solution has the form

$$y(\tau) = C_1 J_{\sqrt{2}}(\tau) + C_2 J_{-\sqrt{2}}(\tau).$$

Returning to the original variable, we obtain

$$y(x) = C_1 J_{\sqrt{2}}(x^2) + C_2 J_{-\sqrt{2}}(x^2).$$

4 Bessel functions of the second kind

As elucidated in the previous section, for integer $\nu = n$ the particular solutions of Bessel's (3.2) equation (3.1)

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \tag{4.1}$$

are linearly dependent since

$$J_{-n}(x) = (-1)^n J_n(x),$$

i.e., in this case we have found only one particular solution of Eq. (4.1).

Consider Bessel's equation with noninteger ν. Its general solution has the form (see Fig. 5)

$$y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x).$$

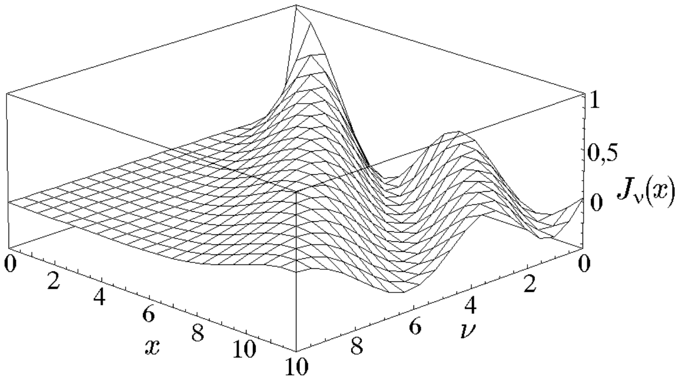

Fig. 5.

Putting

$$C_1 = \operatorname{ctg} \pi\nu, \qquad C_2 = -\frac{1}{\sin \pi\nu},$$

we obtain the function

$$N_\nu(x) = \frac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu}. \tag{4.2}$$

♦ The function (4.2) is called *the Neumann function.*

◊ It was introduced by Weber and sometimes is called the Weber function and often denoted by $Y_\nu(x)$. The function $N_\nu(x)$ is also called the Bessel (or cylindrical) function of the second kind of order ν of argument x (see Fig. 6).

◊ The Neumann function (4.2) is defined for noninteger ν. However, its definition can be extended to integer ν by putting

$$N_n(x) = \lim_{\nu\to n} N_\nu(x).$$

Statement 4.1. *There exists the limit*

$$N_n(x) = \lim_{\nu\to n} N_\nu(x) = \frac{2}{\pi}\Big(\gamma + \ln\frac{x}{2}\Big)J_n(x) - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\Big(\frac{x}{2}\Big)^{-n+2k} -$$

$$-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\Big(\frac{x}{2}\Big)^{2k+n}\Big\{\sum_{m=1}^{k+n}\frac{1}{m} + \sum_{m=1}^{k}\frac{1}{m}\Big\}, \tag{4.3}$$

where $\gamma \approx 0{,}5772\ldots$ is the Euler constant.

The proof of this statement can be found, e.g., in [3].

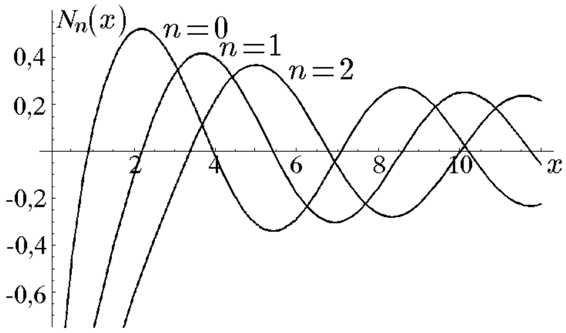

Fig. 6.

Statement 4.2. *The functions $J_\nu(x)$ and $N_\nu(x)$ are linearly independent for any ν.*

To prove that the Neumann function $N_\nu(x)$ and the Bessel function of the first kind $J_\nu(x)$ are linearly independent for any ν, we consider the Wronskian

$$W(x) = W[J_\nu(x), N_\nu(x)] = \begin{vmatrix} J_\nu(x) & N_\nu(x) \\ J'_\nu(x) & N'_\nu(x) \end{vmatrix} = \begin{vmatrix} J_\nu(x) & \dfrac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu} \\ J'_\nu(x) & \dfrac{J'_\nu(x)\cos\pi\nu - J'_{-\nu}(x)}{\sin\pi\nu} \end{vmatrix} =$$

$$= \frac{\cos\pi\nu}{\sin\pi\nu}\begin{vmatrix} J_\nu(x) & J_\nu(x) \\ J'_\nu(x) & J'_\nu(x) \end{vmatrix} - \frac{1}{\sin\pi\nu}\begin{vmatrix} J_\nu & J_{-\nu}(x) \\ J'_\nu(x) & J'_{-\nu}(x) \end{vmatrix} = -\frac{1}{\sin\pi\nu}W[J_\nu(x), J_{-\nu}(x)].$$

From relation (4.2) we obtain

$$W[J_\nu(x), N_\nu(x)] = -\frac{1}{\sin\pi\nu}W[J_\nu(x), J_{-\nu}(x)] = -\frac{1}{\sin\pi\nu}\Big(-\frac{2}{\pi x}\Big)\sin\pi\nu.$$

Thus,

$$W[J_\nu(x), N_\nu(x)] = \frac{2}{\pi x}. \tag{4.4}$$

This formula has been obtained under the assumption that ν is not an integer. However, in view of Theorem 4.1, it can be extended to integer $\nu = n$.

From relation (4.4) it follows that $W[J_\nu(x), N_\nu(x)] \neq 0$ for all values of ν and x. Hence, $J_\nu(x)$ and $N_\nu(x)$ are linearly independent, Q. E. D.

Statement 4.3. *The functions $J_\nu(x)$ and $N_\nu(x)$ form a fundamental set of solutions of Bessel's equation for any, integer included, order. The general solution of Bessel's equation* (3.1) *of any order ν is given by the formula*

$$y(x) = C_1 J_\nu(x) + C_2 N_\nu(x). \tag{4.5}$$

The validity of this statement immediately follows from the fact that the functions $J_\nu(x)$ and $N_\nu(x)$ are linearly independent and are solutions of the second order linear differential equation (3.1).

Example 4.1. Find the general solution of the equation

$$x^2y'' + xy' + \Big(a^2x^2 - \nu^2\Big)y = 0, \tag{4.6}$$

where a and ν are some constants.

Solution. Put $x = t/a$, then

$$\frac{dy}{dx} = a\frac{dy}{dt}, \qquad \frac{d^2y}{dx^2} = a^2\frac{d^2y}{dt^2}$$

and the equation takes the form

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + \Big(t^2 - \nu^2\Big)y = 0.$$

Hence,

$$y(t) = C_1 J_\nu(t) + C_2 N_\nu(t).$$

Returning to the original variable, we obtain

$$y(x) = C_1 J_\nu(ax) + C_2 N_\nu(ax). \tag{4.7}$$

5 Recurrence relations for Bessel functions

Theorem 5.1. *The following recurrence relations are valid:*

$$\frac{d}{dx}[x^{-\nu}J_\nu(x)] = -\frac{J_{\nu+1}(x)}{x^\nu}, \tag{5.1}$$
$$\frac{d}{dx}[x^{\nu}J_\nu(x)] = x^\nu J_{\nu-1}(x). \tag{5.2}$$

Proof. From definition (3.6) we obtain

$$\frac{d}{dx}\Big[\frac{J_\nu(x)}{x^\nu}\Big] = \frac{d}{dx}\Big[\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\frac{x^{2k}}{2^{2k+\nu}}\Big] = \sum_{k=0}^{\infty}\frac{(-1)^k 2k}{k!\Gamma(k+\nu+1)}\frac{x^{2k-1}}{2^{2k+\nu}} =$$
$$= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}(-1)}{(k-1)!\Gamma([k-1]+[\nu+1]+1)}\frac{x^{2(k-1)+1+\nu}}{2^{2(k-1)+(\nu+1)}}\frac{1}{x^\nu} = \frac{(-1)}{x^\nu}J_{\nu+1}(x).$$

Similarly,

$$\frac{d}{dx}[x^\nu J_\nu(x)] = \frac{d}{dx}\Big[\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\frac{x^{2k+2\nu}}{2^{2k+\nu}}\Big] = \sum_{k=0}^{\infty}\frac{(-1)^k 2(k+\nu)}{k!\Gamma(k+\nu+1)}\frac{x^{2k+2\nu-1}}{2^{2k+\nu}} =$$
$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+[\nu-1]+1)}\frac{x^{2k+(\nu-1)}}{2^{2k+(\nu-1)}}x^\nu = x^\nu J_{\nu-1}(x).$$

Corollary 5.1.1. The following recurrence relations are valid:

$$xJ'_\nu(x) = \nu J_\nu(x) - xJ_{\nu+1}(x). \tag{5.3}$$
$$xJ'_\nu(x) = -\nu J_\nu(x) + xJ_{\nu-1}(x). \tag{5.4}$$

Proof immediately follows from relations (5.1) and (5.2).

Corollary 5.1.2. The following recurrence relations are valid:

$$2J'_\nu(x) = J_{\nu-1}(x) - J_{\nu+1}(x), \tag{5.5}$$
$$2\frac{\nu}{x}J_\nu(x) = J_{\nu-1}(x) + J_{\nu+1}(x). \tag{5.6}$$

Proof. Combining relations (5.3) and (5.4), we obtain (5.5). Subtracting (5.4) from (5.3) yields (5.6).

◊ Put $\nu = 0$ in (5.3). Then,

$$J'_0(x) = -J_1(x). \tag{5.7}$$

Hence, the zeros of the function $J_1(x)$ coincide with the maxima and minima of the function $J_0(x)$ (see Fig. 2).

Corollary 5.1.3. The following relations are valid:

$$\Big(\frac{1}{x}\frac{d}{dx}\Big)^m\Big[\frac{J_\nu(x)}{x^\nu}\Big] = (-1)^m\frac{J_{\nu+m}(x)}{x^{\nu+m}}, \tag{5.8}$$
$$\Big(\frac{1}{x}\frac{d}{dx}\Big)^m[x^\nu J_\nu(x)] = x^{\nu-m}J_{\nu-m}(x). \tag{5.9}$$

Proof. Apply the method of mathematical induction. Represent (5.1) as

$$\frac{1}{x}\frac{d}{dx}\left[\frac{J_\nu(x)}{x^\nu}\right] = -\frac{J_{\nu+1}(x)}{x^{\nu+1}}.$$

Thus, for $m = 1$ relation (5.8) holds. Assume that (5.8) is fulfilled for a certain $m = k$. Check the validity of relation (5.8) for $m = k+1$. To do this, differentiate this relation with respect to x and use relation (5.1) to get

$$\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}\right)^k\left[\frac{J_\nu(x)}{x^\nu}\right] = (-1)^k\frac{d}{dx}\left[\frac{J_{\nu+k}(x)}{x^{\nu+k}}\right] = (-1)^{k+1}\frac{J_{\nu+k+1}(x)}{x^{\nu+k}}.$$

Hence,

$$\left(\frac{1}{x}\frac{d}{dx}\right)\left(\frac{1}{x}\frac{d}{dx}\right)^k\left[\frac{J_\nu(x)}{x^\nu}\right] = (-1)^{k+1}\frac{J_{\nu+k+1}(x)}{x^{\nu+k+1}},$$

which proves the statement. Relation (5.9) is proved in a like manner.

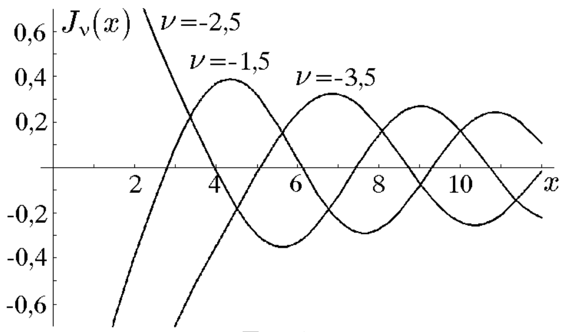

Fig. 7.

Corollary 5.1.4. Bessel functions of half-integer order can be expressed in terms of elementary functions (see Fig. 7)

$$J_{1/2+m}(x) = (-1)^m x^{m+1/2}\left[\frac{1}{x}\frac{d}{dx}\right]^m\left[\sqrt{\frac{2}{\pi x}}\frac{\sin x}{\sqrt{x}}\right]; \tag{5.10}$$

$$J_{-1/2-m}(x) = x^{m+1/2}\left[\frac{1}{x}\frac{d}{dx}\right]^m\left[\sqrt{\frac{2}{\pi x}}\frac{\cos x}{\sqrt{x}}\right]. \tag{5.11}$$

Proof. Calculate $J_{1/2}(x)$ and $J_{-1/2}(x)$:

$$J_{1/2}(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+3/2)}\left(\frac{x}{2}\right)^{2k+1/2};$$

$$J_{-1/2}(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+1/2)}\left(\frac{x}{2}\right)^{2k-1/2}.$$

From the principal functional relation for the gamma-function $\Gamma(z+1) = z\Gamma(z)$ and from the fact that $\Gamma(1/2) = \sqrt{\pi}$ it follows that

$$\Gamma(k+3/2) = \frac{(2k+1)!!}{2^{k+1}}\sqrt{\pi} \quad \text{and} \quad \Gamma(k+1/2) = \frac{(2k-1)!!}{2^k}\sqrt{\pi}.$$

Hence,

$$J_{1/2}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{k+1}}{k!(2k+1)!!\sqrt{\pi}} \Big(\frac{x}{2}\Big)^{2k+1/2}$$

and

$$J_{-1/2}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{k}}{k!(2k-1)!!\sqrt{\pi}} \Big(\frac{x}{2}\Big)^{2k-1/2}.$$

Note that

$$k!(2k+1)!! = \frac{(2k)!!}{2^k}(2k+1)!! = \frac{(2k+1)!}{2^k},$$
$$k!(2k-1)!! = \frac{(2k)!}{2^k}.$$

Then,

$$J_{1/2}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\sqrt{2}}{(2k+1)!\sqrt{\pi}} x^{2k+1} \frac{1}{\sqrt{x}} = \sqrt{\frac{2}{\pi x}} \sin x,$$
$$J_{-1/2}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\sqrt{2}}{(2k)!\sqrt{\pi}} x^{2k} \frac{1}{\sqrt{x}} = \sqrt{\frac{2}{\pi x}} \cos x.$$

Hence,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad \text{and} \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x. \tag{5.12}$$

Putting $\nu = 1/2$ in (5.8) and $\nu = -1/2$ in (5.9) and making use of relations (5.12), we obtain formulas (5.10) and (5.11), Q. E. D.

Example 5.1. Calculate $J_{3/2}(x)$ and $J_{-3/2}(x)$.

Proof. Substitute $m = 1$ into (5.10) and $m = 1$ into (5.11). Then,

$$\begin{aligned} J_{3/2}(x) &= \sqrt{\frac{2}{\pi x}}\Big(\frac{\sin x}{x} - \cos x\Big), \\ J_{-3/2}(x) &= \sqrt{\frac{2}{\pi x}}\Big(-\sin x - \frac{\cos x}{x}\Big). \end{aligned} \tag{5.13}$$

Example 5.2. Prove the validity of the relations

$$\begin{aligned} &J_0'(x) = -J_1(x), \\ &J_2(x) - J_0(x) = 2J_0''(x), \\ &x^2 J_n''(x) = (n^2 - n - x^2)J_n + xJ_{n+1}(x). \end{aligned} \tag{5.14}$$

Solution. Applying the recurrence relation

$$-\frac{1}{x}\frac{d}{dx}[x^{-\nu}J_\nu(x)] = x^{-(\nu+1)}J_{\nu+1}(x)$$

for $\nu = 0$, we obtain the first equality.

Differentiating the obtained equality and using the relation

$$J'_\nu(x) = \frac{1}{2}[J_{\nu-1}(x) - J_{\nu+1}(x)]$$

for $\nu = 1$, we arrive at the second equality.

From the identity

$$x^2 J''_n(x) + xJ'_n(x) + (x^2 - n^2)J_n(x) \equiv 0$$

and the recurrence relation (5.5) it follows that

$$x^2 J''_n(x) = -\frac{x}{2}[J_{n-1}(x) - J_{n+1}(x)] - (x^2 - n^2)J_n(x).$$

Eliminate $J_{n-1}(x)$ from this equality with the help of relation (5.6) to get

$$x^2 J''_n(x) = -\frac{x}{2}\Big[\frac{2n}{x}J_n(x) - 2J_{n+1}(x)\Big] - (x^2 - n^2)J_n(x)$$

or

$$x^2 J''_n(x) = (n^2 - n - x^2)J_n(x) + xJ_{n+1}(x).$$

6 Bessel functions of different kinds

♦ The functions

$$H^{(1)}_\nu(x) = J_\nu(x) + iN_\nu(x), \quad H^{(2)}_\nu(x) = J_\nu(x) - iN_\nu(x) \tag{6.1}$$

are called, respectively, *the first and the second Hankel functions* or *Bessel functions of the third kind.*

Making use of the definitions of a Neumann function $N_\nu(x)$ (4.2), we obtain

$$\begin{aligned} H^{(1)}_\nu(x) &= \frac{J_{-\nu}(x) - e^{-i\pi\nu}J_\nu(x)}{i\sin\pi\nu}, \\ H^{(2)}_\nu(x) &= \frac{-J_{-\nu}(x) + e^{i\pi\nu}J_\nu(x)}{i\sin\pi\nu}. \end{aligned} \tag{6.2}$$

♢ In formulas (6.2), the uncertainty of the type 0/0 appears as $\nu \to n$, which can be removed, for instance, by the L'Hospital rule.

♢ From (6.2) it follows, in particular, that

$$H^{(1)}_{-\nu}(x) = e^{i\pi\nu}H^{(1)}_\nu(x) \qquad \text{and} \qquad H^{(2)}_{-\nu}(x) = e^{-i\pi\nu}H^{(2)}_\nu(x).$$

Example 6.1. Express the Hankel functions $H^{(1,2)}_{1/2}(x)$ and $H^{(1,2)}_{-1/2}(x)$ in terms of elementary functions.

Solution. Since $\sin(\pi/2) = 1$ and $e^{i\pi/2} = i$, from (6.2), in view of (5.12), we obtain

$$\begin{aligned} H^{(1)}_{1/2}(x) &= -i\sqrt{\frac{2}{\pi x}}e^{ix}, \quad H^{(1)}_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}e^{ix}; \\ H^{(2)}_{1/2}(x) &= i\sqrt{\frac{2}{\pi x}}e^{-ix}, \quad H^{(2)}_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}e^{-ix}. \end{aligned}$$

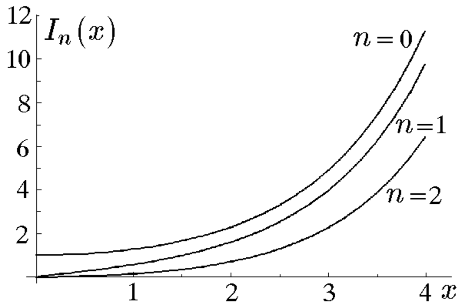

Fig. 8.

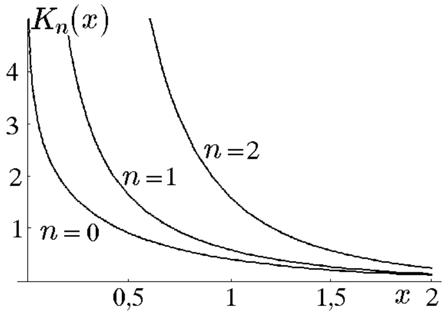

Fig. 9.

◊ Comparison of formulas (6.1) with the Euler formulas shows that $J_\nu(x)$ is analogous to the function $\cos x$, $N_\nu(x)$ to $\sin x$, and $H^{(1)}(x)$ to e^{ix}.

♦ The function

$$I_\nu(x) = i^{-\nu} J_\nu(ix) = e^{-i\pi\nu/2} J_\nu(ix) \tag{6.3}$$

is called *a modified Bessel function of the first kind of order* ν (see Fig. 8).

♦ The function

$$K_\nu(x) = i^{\nu+1}\frac{\pi}{2} H^{(1)}_\nu(ix) \tag{6.4}$$

is called *a modified Bessel function of the second kind* or *the Macdonald function* (see Fig. 9).

◊ From the definition of the function $J_\nu(x)$ it follows that the function $I_\nu(x)$ can be represented by a Taylor series of the form

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu+k+1)} \Big(\frac{x}{2}\Big)^{2k+\nu}. \tag{6.5}$$

◊ The functions $K_\nu(x)$ and $I_\nu(x)$ are related as

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \pi\nu}.$$

Theorem 6.1. *The functions* $I_\nu(x)$ *and* $K_\nu(x)$ *are linearly independent solutions of the equation*

$$x^2 y'' + xy' - (x^2 + \nu^2)y = 0. \tag{6.6}$$

Proof. The change of variables $z = ix$ reduces Eq. (6.6) to Bessel's equation

$$z^2 \frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - \nu^2)y = 0,$$

whose particular solutions are the functions

$$y(z) = C_1 J_\nu(z) = C_1 J_\nu(ix) \quad \text{and} \quad y(z) = C_2 J_{-\nu}(z) = C_2 J_{-\nu}(ix),$$

where C_1, C_2 are arbitrary constants. Choosing, in accordance with (6.3), $C_1 = i^{-\nu}$ and $C_2 = i^\nu$, we make sure of the fact that $I_\nu(x)$ and $I_{-\nu}(x)$ are particular solutions of Eq. (6.6).

The Wronskian of the functions $i^{-\nu} J_\nu(z)$ and $i^\nu J_{-\nu}(z)$, according to (3.10), can be written as

$$W[i^{-\nu} J_\nu(z), i^\nu J_{-\nu}(z)] = W[J_\nu(z), J_{-\nu}(z)] = -\frac{2}{\pi z} \sin \pi\nu$$

or, in expanded form,

$$i^{-\nu}J_\nu(z)\frac{d}{dz}[i^\nu J_{-\nu}(z)] - i^\nu J_{-\nu}(z)\frac{d}{dz}[i^{-\nu}J_\nu(z)] = -\frac{2}{\pi z}\sin\pi\nu.$$

The latter equality, in view of definition (6.3) and the relations

$$z = ix, \qquad \frac{d}{dz} = \frac{1}{i}\frac{d}{dx},$$

takes the form

$$I_\nu(x)\frac{dI_{-\nu}(x)}{dx} - I_{-\nu}(x)\frac{dI_\nu(x)}{dx} = W[I_\nu(x), I_{-\nu}(x)] = -\frac{2}{\pi x}\sin\pi\nu$$

which implies that the functions $I_\nu(x)$ and $I_{-\nu}(x)$ are linearly independent for $\nu \neq n$.

Calculate

$$W[I_\nu(x), K_\nu(x)] = W\Big[I_\nu(x), \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin\pi\nu}\Big] =$$
$$= \frac{\pi}{2\sin\pi\nu}W[I_\nu(x), I_{-\nu}(x)] = \frac{\pi}{2\sin\pi\nu}\Big(-\frac{2}{\pi x}\sin\pi\nu\Big) = -\frac{1}{x}.$$

Hence, $I_\nu(x)$ and $K_\nu(x)$ are linearly independent for any ν, Q. E. D.

Corollary. The functions $I_\nu(x)$ and $K_\nu(x)$ form a fundamental set of solutions of Eq. (6.6) and the general solution of Eq. (6.6) has the form

$$y(x) = C_1 I_\nu(x) + C_2 K_\nu(x). \tag{6.7}$$

Proof follows from the fact that $I_\nu(x)$ and $K_\nu(x)$ are linearly independent for any ν and are solutions of Eq. (6.6).

Theorem 6.2. *For Bessel functions of imaginary argument, the following recurrence relations are valid:*

$$\begin{aligned} I_{\nu-1}(x) - I_{\nu+1}(x) &= \frac{2\nu}{x}I_\nu(x), \\ I_{\nu-1}(x) + I_{\nu+1}(x) &= 2I'_\nu(x). \end{aligned} \tag{6.8}$$

Proof. Changing x by ix in the recurrence relation

$$J_{\nu+1}(x) + J_{\nu-1}(x) = \frac{2\nu}{x}J_\nu(x)$$

and multiplying this equality by $i^{-(\nu+1)}$, we obtain the first formula. To derive the second formula, we use the recurrence relation

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x).$$

Example 6.2. Find the general solution of the equation

$$y'' + \Big(4 + \frac{1}{x}\Big)y' + \Big(3 + \frac{2}{x} - \frac{5}{x^2}\Big)y = 0. \tag{6.9}$$

Solution. Make the change of variables

$$y(x) = e^{-2x}z(x). \tag{6.10}$$

Calculate the derivatives:

$$\begin{aligned} y'(x) &= [e^{-2x}z(x)]' = e^{-2x}[-2z(x) + z'(x)]; \\ y''(x) &= \{e^{-2x}[-2z(x) + z'(x)]\}' = e^{-2x}[z''(x) - 4z'(x) + 4z(x)]. \end{aligned} \tag{6.11}$$

Substitute (6.10) and (6.11) into (6.9)

$$e^{-2x}[z''(x)-4z'(x)+4z(x)]+\Big(4+\frac{1}{x}\Big)e^{-2x}[z'(x)-2z(x)]+\Big(3+\frac{2}{x}-\frac{5}{x^2}\Big)e^{-2x}z(x)=0.$$

Dividing this by the exponential function and grouping similar terms, we obtain

$$x^2z''(x)+xz'(x)+(-x^2-5)z(x)=0. \tag{6.12}$$

Equation (6.12) is Bessel's equation (6.6) with $\nu=\sqrt{5}$ whose general solution, according to (6.7), has the form

$$z(x)=C_1I_{\sqrt{5}}(x)+C_2K_{\sqrt{5}}(x).$$

Finally, we obtain

$$y(x)=C_1e^{-2x}I_{\sqrt{5}}(x)+C_2e^{-2x}K_{\sqrt{5}}(x)$$

7 The asymptotic behavior of cylindrical functions

Theorem 7.1. *The function*

$$y(x)=\frac{A}{\sqrt{x}}\cos(x+\varphi), \tag{7.1}$$

where A and φ are arbitrary constants, is an asymptotic solution of Bessel's equation (3.1) *for any ν up to $O(x^{-3/2})$ as $x\to\infty$.*

Proof. Consider Bessel's equation

$$y''+\frac{1}{x}y'+\Big(1-\frac{\nu^2}{x^2}\Big)y=0. \tag{7.2}$$

Make in Eq. (7.2) the change of variables $x=\varepsilon t$, $t\in]0,1]$, $\varepsilon=\text{const}$. Then for large x, i.e., as $x\to\infty$, the parameter ε will tend to zero as well.

Equation (7.2) in new variables takes the form

$$\frac{1}{\varepsilon^2}\ddot y+\frac{1}{\varepsilon^2 t}\dot y+\Big(1-\frac{\nu^2}{\varepsilon^2t^2}\Big)y=0. \tag{7.3}$$

Here, the dot denotes the differentiation with respect to the variable t, $\dot y=dy/dt$.

We seek the solution of Eq. (7.3) in the form

$$y(t,\varepsilon)=e^{i\varepsilon S(t)}f(t,\varepsilon), \tag{7.4}$$

where the function $S(t)$ does not depend on ε, the function $f(t,\varepsilon)$ depends regularly on the parameter $\omega=1/\varepsilon$, i.e.,

$$f(t,\varepsilon)=f_0(t)+\frac{1}{\varepsilon}f_1(t)+\frac{1}{\varepsilon^2}f_2(t)+\dots, \tag{7.5}$$

and $f_k(t)$, $k=\overline{0,\infty}$, are independent of ε. From (7.4) we find

$$\begin{aligned}\dot y(t,\varepsilon)&=\dot f(t,\varepsilon)e^{i\varepsilon S(t)}+i\varepsilon\dot S(t)f(t,\varepsilon)e^{i\varepsilon S(t)},\\ \ddot y(t,\varepsilon)&=\ddot f(t,\varepsilon)e^{i\varepsilon S(t)}+2i\varepsilon\dot S(t)\dot f(t,\varepsilon)e^{i\varepsilon S(t)}-\varepsilon^2(\dot S(t))^2f(t,\varepsilon)e^{i\varepsilon S(t)}.\end{aligned} \tag{7.6}$$

Substituting (7.4) and (7.6) into Eq. (7.3), we obtain

$$\Big\{[-\dot{S}^2(t)+1]f(t,\varepsilon)+\frac{1}{\varepsilon}\Big[2i\dot{f}(t,\varepsilon)\dot{S}(t)+\frac{i}{t}f(t,\varepsilon)\dot{S}(t)\Big]+ \\ +\frac{1}{\varepsilon^2}\Big[\ddot{f}(t,\varepsilon)+\frac{1}{t}\dot{f}(t,\varepsilon)-\frac{\nu^2}{t^2}f(t,\varepsilon)\Big]\Big\}e^{i\varepsilon S(t)}=0.$$

Substitute the expansion (7.5) into this equation and equate coefficients of terms with like powers of $1/\varepsilon$ to get

$$\begin{array}{l|l} \varepsilon^0 & 1-\dot{S}^2(t)=0 \\ \varepsilon^{-1} & 2\dot{f}_0+\dfrac{1}{t}f_0=0 \\ \dots & \dots\dots\dots\dots\dots \end{array}$$

From these equations we find

$$S(t)=\pm t, \qquad f_0(t)=\frac{C}{\sqrt{t}}.$$

Hence, the function

$$y(t,\varepsilon)=\frac{C_\pm}{\sqrt{t}}\exp(\pm i\varepsilon t) \tag{7.7}$$

is an asymptotic solution of Eq. (7.3), up to $O(1/\varepsilon)$ as $\varepsilon\to 0$. Equation (7.3) is a second order linear differential equation with real coefficients. Hence, the real and the imaginary part of its complex solutions are solutions of Eq. (7.3) as well. Thus, the function

$$y(t,\varepsilon)=\frac{C_1}{\sqrt{t}}\cos\varepsilon t+\frac{C_2}{\sqrt{t}}\sin\varepsilon t \tag{7.8}$$

is an asymptotic solution of Eq. (7.3) as well. Putting in (7.8)

$$C_1=\frac{A}{\sqrt{\varepsilon}}\cos\varphi, \qquad C_2=\frac{A}{\sqrt{\varepsilon}}\sin\varphi$$

and returning to the original variables, we come to the statement of the theorem.

It can be shown that as $x\to\infty$ the following estimates are valid:

$$J_\nu(x)=\sqrt{\frac{2}{\pi x}}\cos\Big(x-\frac{\pi\nu}{2}-\frac{\pi}{4}\Big)+O(x^{-3/2}), \tag{7.9}$$

$$N_\nu(x)=\sqrt{\frac{2}{\pi x}}\sin\Big(x-\frac{\pi\nu}{2}-\frac{\pi}{4}\Big)+O(x^{-3/2}), \tag{7.10}$$

$$H_\nu(x)=\sqrt{\frac{2}{\pi x}}\exp\Big[\pm i\Big(x-\frac{\pi\nu}{2}-\frac{\pi}{4}\Big)\Big]+O(x^{-3/2}). \tag{7.11}$$

$\Diamond$ Note that the asymptotic estimates (7.9)–(7.11) are valid not only for $x\gg 1$, but also for all $|z|\gg 1$, $\arg z<\pi-\delta$ with δ being an arbitrary small positive number.

◊ Similarly, an asymptotic expansion can be obtained for Bessel functions of imaginary argument as $x \to \infty$:

$$I_n(x) = \sqrt{\frac{1}{2\pi x}} e^x [1 + O(1/x)], \tag{7.12}$$

$$K_n(x) = \sqrt{\frac{1}{2x}} e^{-x} [1 + O(1/x)]. \tag{7.13}$$

Consider the behavior of cylindrical functions as $x \to +0$. From definition (3.6) it follows that

$$J_\nu(x) = \frac{1}{\Gamma(\nu+1)} \Big(\frac{x}{2}\Big)^\nu + O(x^\nu), \quad x \to +0, \ \nu \geqslant 0. \tag{7.14}$$

From definition (4.2) and relation (4.3) we get

$$N_\nu(x) = -\frac{\Gamma(\nu)}{\pi} \Big(\frac{x}{2}\Big)^{-\nu} + O(x^{-\nu}), \quad x \to 0, \ \nu > 0,$$

$$N_0(x) = \frac{2}{\pi} \ln \frac{x}{2} + O(1), \quad x \to 0.$$

From (6.2) we find

$$H_0^{(1,2)}(x) = \pm i \ln \frac{x}{2},$$

$$H_\nu^{(1,2)}(x) = \pm i \frac{\Gamma(\nu)}{\pi} \Big(\frac{x}{2}\Big)^{-\nu}, \quad \nu > 0, \ x \to +0.$$

Similarly, for Bessel functions of imaginary argument we have

$$I_\nu(x) = \frac{1}{\Gamma(\nu+1)} \Big(\frac{x}{2}\Big)^\nu + O(x^\nu), \quad x \to +0, \ \nu \geqslant 0;$$

$$K_\nu(x) = \frac{1}{2}\Gamma(\nu) \Big(\frac{x}{2}\Big)^{-\nu} + O(x^{-\nu}), \quad x \to +0, \ \nu > 0;$$

$$K_0(x) = \ln \frac{x}{2} + O(1), \quad x \to 0.$$

Example 7.1. Prove that

$$\int\limits_0^\infty J_n(x)dx = \int\limits_0^\infty J_{n+2}(x)dx, \quad n = \overline{0, \infty}. \tag{7.15}$$

Solution. Consider the difference $J_n(x) - J_{n+2}(x)$. From formula (5.5) for $\nu = n+1$ we find

$$J_n(x) - J_{n+2}(x) = 2J'_{n+1}(x).$$

Then,

$$\int\limits_0^\infty [J_n(x) - J_{n+2}(x)]dx = 2\int\limits_0^\infty J'_{n+1}(x)dx = 2J_{n+1}(x)\Big|_0^\infty.$$

For Bessel functions, the asymptotic estimate

$$J_\nu(x) \approx \Big(\frac{2}{\pi x}\Big)^{1/2} \cos\Big(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big), \quad x \to \infty \tag{7.16}$$

is valid for large x. Hence,

$$\lim_{x\to\infty} J_\nu(x) = 0. \tag{7.17}$$

Taking into account that $J_{n+1}(0) = 0$ for $n = \overline{0,\infty}$, we obtain

$$\int_0^\infty [J_n(x) - J_{n+2}(x)]dx = 0 \text{ or } \int_0^\infty J_n(x)dx = \int_0^\infty J_{n+2}(x)dx,$$

Q. E. D.

8 Generating functions. The Bessel integral

♦ The function $\Phi(z,t)$ expandable in a converging series in t

$$\Phi(z,t) = \sum_{n=-\infty}^{\infty} a_n(z)t^n \tag{8.1}$$

with coefficients $a_n(z)$ is called the generating function of a functional sequence $\{a_n(z)\}$, $n = \overline{-\infty,\infty}$.

Theorem 8.1. *The function*

$$F(z,t) = \exp\Big[\frac{1}{2}z\Big(t - \frac{1}{t}\Big)\Big] \tag{8.2}$$

is generating for Bessel functions of integer order, i.e.,

$$\exp\Big[\frac{1}{2}z\Big(t - \frac{1}{t}\Big)\Big] = \sum_{n=-\infty}^{\infty} J_n(z)t^n. \tag{8.3}$$

Proof. Consider the function of complex variable t (8.2). It has essentially singular points $t = 0$ and $t = \infty$ and, hence, is expandable in a Laurent series on the complex plane t with the series coefficients being functions of the parameter z involved in relation (8.2), i.e.,

$$F(z,t) = \sum_{n=-\infty}^{\infty} a_n(z)t^n, \tag{8.4}$$

where

$$a_n = \frac{1}{2\pi i}\int_{\gamma_0} \frac{\exp[(z/2)(\eta - 1/\eta)]}{\eta^{n+1}}d\eta. \tag{8.5}$$

Here, γ_0 is an arbitrary closed contour enclosing the origin.

Perform the change of variables $\eta = 2t/z$, $(z \neq 0)$, with $\gamma_0 \to \gamma$, γ also enclosing the origin. Then,

$$a_n(z) = \frac{1}{2\pi i}\Big(\frac{z}{2}\Big)^n \int_\gamma t^{-n-1}e^{t-z^2/4t}dt,$$

but

$$e^{-z^2/4t} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\frac{z^{2k}}{2^{2k}t^k},$$

with the series converging uniformly for all $|z| < \infty$. Then,

$$a_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\Big(\frac{z}{2}\Big)^{2k+n}\Big\{\frac{1}{2\pi i}\int\limits_{\gamma} t^{-n-1-k}e^t dt\Big\}.$$

If $n+k<0$, the integrand is an analytic function and the integral is equal to zero. If $n+k \geqslant 0$, the point $t=0$ is a pole of order $n+k+1$. Hence,

$$\frac{1}{2\pi i}\int\limits_{\gamma} t^{-n-1-k}e^t dt = \lim_{t\to 0}\frac{1}{(n+k)!}\frac{d^{n+k}}{dt^{n+k}}\Big[\frac{e^t}{t^{n+k+1}}t^{n+k+1}\Big].$$

Since

$$\frac{1}{2\pi i}\int\limits_{\gamma} t^{-n-1-k}e^t dt = \begin{cases}\dfrac{1}{(n+k)!}, & n+k \geqslant 0;\\ 0, & k+n<0\end{cases} = \frac{1}{\Gamma(n+k+1)},$$

we get

$$a_n(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+n+1)}\Big(\frac{z}{2}\Big)^{2k+n} = J_n(z).$$

Corollary 8.1.1. The formula

$$e^{-iz\sin\varphi} = \sum_{n=-\infty}^{\infty} J_n(z)e^{-in\varphi} \tag{8.6}$$

is valid.

Proof. Put in formula (8.3) $t = e^{-i\varphi}$ to get

$$F(z, e^{i\varphi}) = e^{-iz\sin\varphi} = \sum_{n=-\infty}^{\infty} J_n(z)e^{-in\varphi},$$

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Corollary 8.1.2. The relations

$$J_n(z) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} e^{i(n\varphi - z\sin\varphi)}d\varphi = \frac{1}{\pi}\int\limits_{0}^{\pi}\cos(n\varphi - z\sin\varphi)d\varphi, \quad n=\overline{0,\infty} \tag{8.7}$$

are valid. Formula (8.7) is called the Bessel integral or, with $n=0$, the Parseval integral.

Proof. Expand the function $e^{-iz\sin\varphi}$ in a Fourier series in variable φ:

$$e^{-iz\sin\varphi} = \sum_{n=-\infty}^{\infty} C_n e^{-in\varphi}, \tag{8.8}$$

where

$$C_n = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} e^{-iz\sin\varphi + in\varphi}d\varphi.$$

An expansion of a continuous function in a Fourier series is unique. Hence, the coefficients of the series (8.6) and (8.8) are identical and relations (8.7) hold.

◊ The Bessel formula (8.7) is valid only for integer n. Otherwise it has the form

$$J_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(\nu\varphi - z\sin\varphi)d\varphi - \\ -\frac{\sin\pi\nu}{\pi}\int_0^\infty e^{-\nu\varphi - z\,\mathrm{sh}\,\varphi}d\varphi \tag{8.9}$$

(see, e.g., [3]).

Corollary 8.1.3. The relations

$$\cos(z\sin\varphi) = J_0(z) + 2\sum_{n=1}^\infty J_{2n}(z)\cos(2n\varphi); \tag{8.10}$$

$$\sin(z\sin\varphi) = 2\sum_{n=1}^\infty J_{2n-1}(z)\sin(2n-1)\varphi; \tag{8.11}$$

$$\cos(z\cos\theta) = J_0(z) + 2\sum_{n=1}^\infty (-1)^n J_{2n}(z)\cos(2n\theta); \tag{8.12}$$

$$\sin(z\sin\theta) = 2\sum_{n=1}^\infty (-1)^{n+1} J_{2n-1}(z)\cos(2n-1)\theta \tag{8.13}$$

are valid. The expansions (8.10)–(8.13) are called the Jacobi expansions.

Proof. Equating the real and the imaginary parts on both sides of relation (8.6), find (8.10) and (8.10). (We leave this to be done by the student.) Make in (8.10) and (8.10) the change $\varphi = \pi/2 - \theta$ to get (8.12) and (8.13).

From (8.10) and (8.11) it follows that

$$\int_0^\pi \cos(z\sin\varphi)\cos 2n\varphi\, d\varphi = \pi J_{2n}(z), \tag{8.14}$$

$$\int_0^\pi \sin(z\sin\varphi)\sin(2n-1)\varphi\, d\varphi = \pi J_{2n-1}(z). \tag{8.15}$$

Example 8.1. Expand the functions $\cos x$ and $\sin x$ in a series in Bessel functions of integer orders.

Solution. Put $z = x$ in equalities (8.10)–(8.13) to get

$$\cos(x\sin\varphi) = J_0(x) + 2\sum_{k=1}^\infty J_{2k}(x)\cos 2k\varphi,$$

$$\sin(x\sin\varphi) = 2\sum_{k=1}^\infty J_{2k-1}(x)\sin(2k-1)\varphi.$$

For $\varphi = \pi/2$ we finally obtain

$$\cos x = J_0(x) + 2\sum_{k=1}^{\infty} J_{2k}(x)\cos k\pi = J_0(x) + 2\sum_{k=1}^{\infty}(-1)^k J_{2k}(x),$$

$$\sin x = 2\sum_{k=1}^{\infty} J_{2k-1}(x)\sin(2k-1)\frac{\pi}{2} = 2\sum_{k=1}^{\infty}(-1)^{k+1} J_{2k-1}(x).$$

9 Lommel's integrals

Theorem 9.1. *Provided that* $\operatorname{Re}\nu > -1$, *the relations*

$$\int_0^{\zeta} zJ_\nu(\alpha z)J_\nu(\beta z)dz =$$

$$= \frac{\zeta}{\alpha^2-\beta^2}\{\beta J_\nu(\alpha\zeta)J'_\nu(\beta\zeta) - \alpha J_\nu(\beta\zeta)J'_\nu(\alpha\zeta)\}; \tag{9.1}$$

$$= \frac{\zeta}{\alpha^2-\beta^2}\{\alpha J_\nu(\beta\zeta)J_{\nu+1}(\alpha\zeta) - \beta J_\nu(\alpha\zeta)J_{\nu+1}(\beta\zeta)\}; \tag{9.2}$$

$$= \frac{\zeta}{\alpha^2-\beta^2}\{\beta J_{\nu-1}(\beta\zeta)J_\nu(\alpha\zeta) - \alpha J_\nu(\alpha\zeta)J_{\nu-1}(\beta\zeta); \tag{9.3}$$

$$\int_0^{\zeta} zJ_\nu^2(\alpha z)dz = \frac{\zeta^2}{2}\Big\{[J'_\nu(\alpha\zeta)]^2 + \Big(1 - \frac{\nu^2}{\alpha^2\zeta^2}\Big)[J_\nu(\alpha\zeta)]^2\Big\} \tag{9.4}$$

are valid. Relations (9.1)–(9.4) *are called Lommel's integrals.*

Proof. 1. Consider the equation

$$z^2y'' + zy' + (\alpha^2z^2 - \nu^2)y = 0. \tag{9.5}$$

Make in this equation the change of variables $\alpha^2z^2 = t^2$. Then,

$$y' = \frac{dy}{dz} = \alpha\frac{dy}{dt} \qquad \text{and} \qquad y'' = \frac{d^2y}{dz^2} = \alpha^2\frac{d^2y}{dt^2}.$$

As a result we get

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0,$$

and, hence,

$$y(t) = C_1J_\nu(t) + C_2N_\nu(t) = C_1J_\nu(\alpha z) + C_2N_\nu(\alpha z) \tag{9.6}$$

is the general solution of Eq. (9.5). In particular, putting $C_2 = 0$ and $C_1 = 1$, we obtain

$$z^2\frac{d^2J_\nu(\alpha z)}{dz^2} + z\frac{dJ_\nu(\alpha z)}{dz} + (\alpha^2z^2 - \nu^2)J_\nu(\alpha z) = 0$$

and

$$z^2\frac{d^2J_\nu(\beta z)}{dz^2} + z\frac{dJ_\nu(\beta z)}{dz} + (\beta^2z^2 - \nu^2)J_\nu(\beta z) = 0.$$

Multiply the first equation by $\frac{1}{z}J_\nu(\beta z)$ and the second one by $\frac{1}{z}J_\nu(\alpha z)$ and subtract the second from the first. Then,

$$zJ_\nu(\beta z)\frac{d^2J_\nu(\alpha z)}{dz^2} - zJ_\nu(\alpha z)\frac{d^2J_\nu(\beta z)}{dz^2} +$$
$$+J_\nu(\beta z)\frac{dJ_\nu(\alpha z)}{dz} - J_\nu(\alpha z)\frac{dJ_\nu(\beta z)}{dz} = -(\alpha^2-\beta^2)zJ_\nu(\alpha z)J_\nu(\beta z).$$

The left side of this equality can be represented as

$$\frac{d}{dz}\Big[zJ_\nu(\beta z)\frac{dJ_\nu(\alpha z)}{dz} - zJ_\nu(\alpha z)\frac{dJ_\nu(\beta z)}{dz}\Big] = -(\alpha^2-\beta^2)zJ_\nu(\alpha z)J_\nu(\beta z).$$

Integrate the left and the right sides of the equality in the limits from 0 to ζ:

$$-(\alpha^2-\beta^2)\int\limits_0^\zeta zJ_\nu(\alpha z)J_\nu(\beta z)dz = \Big[zJ_\nu(\beta z)\frac{dJ_\nu(\alpha z)}{dz} - zJ_\nu(\alpha z)\frac{dJ_\nu(\beta z)}{dz}\Big]\Big|_0^\zeta =$$
$$= \zeta\Big[J_\nu(\beta\zeta)\frac{dJ_\nu(\alpha\zeta)}{d\zeta} - J_\nu(\alpha\zeta)\frac{dJ_\nu(\beta\zeta)}{d\zeta}\Big] - \lim_{z\to 0} z\Big[J_\nu(\beta z)\frac{dJ_\nu(\alpha z)}{dz} - J_\nu(\alpha z)\frac{dJ_\nu(\beta z)}{dz}\Big].$$

2. Show that this limit is zero. Indeed,

$$J_\nu(\alpha z) = \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(n+\nu+1)}\Big(\frac{\alpha z}{2}\Big)^{2n+\nu} =$$
$$= \frac{1}{\Gamma(\nu+1)}\Big(\frac{\alpha z}{2}\Big)^\nu + \Big(\frac{\alpha z}{2}\Big)^{\nu+2}\sum_{n=1}^\infty \frac{(-1)^n}{n!\Gamma(n+\nu+1)}\Big(\frac{\alpha z}{2}\Big)^{2n-2} =$$
$$= \frac{1}{\Gamma(\nu+1)}\Big(\frac{\alpha z}{2}\Big)^\nu + \Big(\frac{\alpha z}{2}\Big)^{\nu+2}Q_1(\alpha z),$$

where

$$Q_1(\alpha z) = \sum_{n=1}^\infty \frac{(-1)^n}{n!\Gamma(n+\nu+1)}\Big(\frac{\alpha z}{2}\Big)^{2n-2}$$

is a series in positive powers of z. Similarly,

$$\frac{z\,dJ_\nu(\beta z)}{dz} = z\Big[\sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(n+\nu+1)}\Big(\frac{\beta z}{2}\Big)^{2n+\nu}\Big]' =$$
$$= z\sum_{n=0}^\infty \frac{(-1)^n(2n+\nu)}{n!\Gamma(n+\nu+1)}\Big(\frac{\beta z}{2}\Big)^{2n+\nu-1}\frac{\beta}{2} =$$
$$= \frac{\nu}{\Gamma(\nu+1)}\Big(\frac{\beta z}{2}\Big)^\nu + \Big(\frac{\beta z}{2}\Big)^{\nu+2}\sum_{n=1}^\infty \frac{(-1)^n(2n+\nu)}{n!\Gamma(n+\nu+1)}\Big(\frac{\beta z}{2}\Big)^{2n-2} =$$
$$= \frac{\nu}{\Gamma(\nu+1)}\Big(\frac{\beta z}{2}\Big)^\nu + \Big(\frac{\beta z}{2}\Big)^{\nu+2}R_1(\beta z),$$

where

$$R_1(\beta z) = \sum_{n=1}^\infty \frac{(-1)^n(2n+\nu)\beta}{n!\Gamma(n+\nu+1)}\Big(\frac{\beta z}{2}\Big)^{2n-2}$$

is a series in positive powers of z. Then,

$$\lim_{z\to 0} z\Big[J_\nu(\beta z)\frac{dJ_\nu(\alpha z)}{dz} - J_\nu(\alpha z)\frac{dJ_\nu(\beta z)}{dz}\Big] =$$

$$= \lim_{z\to 0}\Big\{\Big[\frac{1}{\Gamma(\nu+1)}\Big(\frac{\beta z}{2}\Big)^\nu + \Big(\frac{\beta z}{2}\Big)^{\nu+2} Q_1(\beta z)\Big]\Big[\frac{\nu}{\Gamma(\nu+1)}\Big(\frac{\alpha z}{2}\Big)^\nu + \Big(\frac{\alpha z}{2}\Big)^{\nu+2} R_1(\alpha z)\Big] -$$

$$- \Big[\frac{1}{\Gamma(\nu+1)}\Big(\frac{\alpha z}{2}\Big)^\nu + \Big(\frac{\alpha z}{2}\Big)^{\nu+2} Q_1(\alpha z)\Big]\Big[\frac{\nu}{\Gamma(\nu+1)}\Big(\frac{\beta z}{2}\Big)^\nu + \Big(\frac{\beta z}{2}\Big)^{\nu+2} R_1(\beta z)\Big]\Big\} = 0.$$

Here we have made use of the fact that $\mathrm{Re}\,\nu > -1$ and $\lim\limits_{z\to 0} Q_1(\alpha z) = \lim\limits_{z\to 0} R_1(\beta z) = 0$ since $Q_1(\alpha z)$ and $R_1(\beta z)$ are series in positive powers of z.

3. Note that

$$\frac{dJ_\nu(\beta\zeta)}{d\zeta} = \beta\frac{dJ_\nu(\beta\zeta)}{d(\beta\zeta)} = \beta J'_\nu(\beta\zeta).$$

Then, for $\alpha \neq \beta$ we have

$$\int_0^\zeta zJ_\nu(\alpha z)J_\nu(\beta z)dz = \frac{\zeta}{\alpha^2-\beta^2}[\alpha J_\nu(\beta\zeta)J'_\nu(\alpha\zeta) - \beta J_\nu(\alpha\zeta)J'_\nu(\beta\zeta)],$$

and thus relation (9.1) is proved. Use in this equality the recurrence relation

$$J'_\nu(\alpha\zeta) = \frac{\nu}{\alpha\zeta}J_\nu(\alpha\zeta) - J_{\nu+1}(\alpha\zeta)$$

and a similar relation for $J'_\nu(\beta\zeta)$ to get (9.2).

Using the other recurrence relation,

$$J'_\nu(\alpha\zeta) = -\frac{\nu}{\alpha\zeta}J_\nu(\alpha\zeta) + J_{\nu-1}(\alpha\zeta),$$

yields (9.3).

4. Lommel's integral (9.4) can be calculated from (9.1) by letting $\beta \to \alpha$. Indeed,

$$\int_0^\zeta zJ_\nu^2(\alpha z)dz = \lim_{\beta\to\alpha}\Big\{\frac{\zeta}{\alpha^2-\beta^2}\big[\beta J_\nu(\alpha\zeta)J'_\nu(\beta\zeta) - \alpha J_\nu(\beta\zeta)J'_\nu(\alpha\zeta)\big]\Big\}.$$

Remove the singularity of type 0/0 appearing in the right side of this expression by the L'Hospital rule:

$$\int_0^\zeta zJ_\nu^2(\alpha z)dz = \lim_{\beta\to\alpha}\frac{\frac{d}{d\beta}\big\{\zeta\big[\beta J_\nu(\alpha\zeta)J'_\nu(\beta\zeta) - \alpha J_\nu(\beta\zeta)J'_\nu(\alpha\zeta)\big]\big\}}{\frac{d}{d\beta}(\alpha^2-\beta^2)} =$$

$$= \lim_{\beta\to\alpha}\frac{\zeta}{2\beta}\big[\alpha\zeta J'_\nu(\beta\zeta)J'_\nu(\alpha\zeta) - J_\nu(\alpha\zeta)J'_\nu(\beta\zeta) - \beta\zeta J_\nu(\alpha\zeta)J''_\nu(\beta\zeta)\big].$$

At the same time, the function $J_\nu(\beta\zeta)$ satisfies the equation

$$(\beta\zeta)^2 J''_\nu(\beta\zeta) + \beta\zeta J'_\nu(\beta\zeta) + (\beta^2\zeta^2 - \nu^2)J_\nu(\beta\zeta) = 0.$$

Then,

$$J''_\nu(\beta\zeta) = -\Big(1 - \frac{\nu^2}{\beta^2\zeta^2}\Big) J_\nu(\beta\zeta) - \frac{1}{\beta\zeta} J'_\nu(\beta\zeta),$$

and the original relation takes the form

$$\int\limits_0^\zeta z J_\nu^2(\alpha z)dz = \lim_{\beta\to\alpha} \frac{\zeta^2}{2\beta}\Big\{\alpha J'_\nu(\alpha\zeta)J'_\nu(\beta\zeta) + \beta\Big[1 - \frac{\nu^2}{\beta^2\zeta^2}\Big] J_\nu(\beta\zeta)J_\nu(\alpha\zeta)\Big\} =$$

$$= \frac{\zeta^2}{2}\Big\{\big[J'_\nu(\alpha\zeta)\big]^2 + \Big[1 - \frac{\nu^2}{\alpha^2\zeta^2}\Big] J_\nu(\alpha\zeta)\Big\},$$

identical to (9.4). Thus, the theorem is proved.

10 Roots of Bessel functions

The problem of finding the roots of Bessel functions plays an important role in applications.

Theorem 10.1. *If the order ν is real and $\nu > -1$, then the function $J_\nu(z)$ has only real roots, that is, if*

$$J_\nu(\alpha_k^\nu) = 0, \qquad \textit{then} \qquad (\alpha_k^\nu)^* = \alpha_k^\nu. \tag{10.1}$$

Proof. We use the rule of contraries.

1. By definition,

$$J_\nu(z) = \Big(\frac{z}{2}\Big)^\nu \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(\nu+k+1)}\Big(\frac{z}{2}\Big)^{2k}. \tag{10.2}$$

Assume that there are pure imaginary roots, i.e., $z = it$. Then we obtain

$$J_\nu(it) = \Big(\frac{i}{2}\Big)^\nu t^\nu \sum_{k=0}^\infty \frac{1}{k!\Gamma(\nu+k+1)}\Big(\frac{t}{2}\Big)^{2k},$$

where only positive summands are under the summation sign. Hence, $J_\nu(it) \neq 0$ for real t other than zero. Thus, the function $J_\nu(z)$ has no pure imaginary roots.

2. Now assume that $\alpha = a + ib$ is a complex root of Eq. (10.1) $J_\nu(\alpha) = 0$ and $a \neq 0$. Since $J_\nu(z)$ is expandable in the power series (10.2) with real coefficients, α^* is also a root of this equation, i.e., $J_\nu(\alpha^*) = 0$.

Now we make use of Lommel's integral (9.1), putting $\alpha = \alpha$ and $\beta = \alpha^*$:

$$\int\limits_0^\zeta z J_\nu(\alpha z)J_\nu(\alpha^* z)dz = \frac{\zeta}{\alpha^2 - (\alpha^*)^2}\{\alpha^* J_\nu(\alpha\zeta)J'_\nu(\alpha^*\zeta) - \alpha J_\nu(\alpha^*\zeta)J'_\nu(\alpha\zeta)\}.$$

Put here $\zeta = 1$. Then, in view of $J_\nu(\alpha) = J_\nu(\alpha^*) = 0$, we obtain

$$\int\limits_0^1 z J_\nu(\alpha z)J_\nu(\alpha^* z)dz = 0. \tag{10.3}$$

Since $J_\nu(\alpha^* z) = [J_\nu(\alpha z)]^*$, then

$$\int_0^1 z J_\nu(\alpha z) J_\nu(\alpha^* z) dz = \int_0^1 z |J_\nu(\alpha z)|^2 dz > 0, \tag{10.4}$$

since the integrand is positive. The resulting contradiction between (10.3) and (10.4) shows that the function $J_\nu(z)$ has no complex roots for $\nu > -1$.

Theorem 10.2. *All roots of the function $J_\nu(z)$, except for $z = 0$, are prime.*

Proof. Assume that $z_0 \neq 0$ is a root of multiplicity $n > 1$. Then, $J_\nu(z_0) = J'_\nu(z_0) = 0$, but

$$J''(z_0) = -\frac{1}{z_0} J'_\nu(z_0) - \Big(1 - \frac{\nu^2}{z_0^2}\Big) J_\nu(z_0) = 0.$$

Differentiating Bessel's equation yields

$$J'''(z_0) = \Big\{\frac{d}{dz}\Big[-\frac{1}{z} J'_\nu(z) - \Big(1 - \frac{\nu^2}{z^2}\Big) J_\nu(z)\Big]\Big\}\Big|_{z=z_0} = 0.$$

Similarly, it can be shown that all derivatives of the function $J_\nu(z)$ at the point z_0 vanish. Hence, $J_\nu(z) \equiv 0$, because all coefficients of the Taylor series at the point z_0 are equal to zero, and $J_\nu(z)$ can not have multiple roots, Q. E. D.

Theorem 10.3. *Equation $J_\nu(z) = 0$ has an infinite set of solutions (roots).*

Proof. The Bessel function $J_\nu(x)$ is a solution of Bessel's equation and therefore it admits, for large x, the following asymptotic representation (7.1)

$$y(x) = \frac{A}{\sqrt{x}} \cos(x + \varphi), \tag{10.5}$$

where $A = \sqrt{2/\pi}$ and $\varphi = -\pi\nu/2$, and the function $\cos x$ has an infinite number of zeros. Hence, $J_\nu(x)$ also has an infinite number of zeros, and the theorem has been proved.

The following theorem describing the behavior of the roots of Bessel functions can be proved (see [21]).

Theorem 10.4 (Bourget). *The functions $J_\nu(z)$ and $J_{\nu+m}(z)$, $m = \overline{1, \infty}$, have no common roots, except for $z = 0$.*

11 Fourier–Bessel and Dini series

Here we consider Fourier–Bessel and Dini series which are important in applications.

Theorem 11.1. *The sequence of functions*

$$J_\nu\Big(\alpha_1^\nu \frac{x}{l}\Big), J_\nu\Big(\alpha_2^\nu \frac{x}{l}\Big), \ldots, J_\nu\Big(\alpha_k^\nu \frac{x}{l}\Big) \ldots, \qquad \nu > -1, \tag{11.1}$$

where α_k^ν, $k = \overline{1, \infty}$ are roots of the equation

$$J_\nu(x) = 0, \tag{11.2}$$

form an orthogonal system of functions with the weight function $\rho(x) = x$*, such that*

$$\int_0^l xJ_\nu\Big(\alpha_k^\nu \frac{x}{l}\Big)J_\nu\Big(\alpha_j^\nu \frac{x}{l}\Big)dx = \frac{l^2}{2}[J'_\nu(\alpha_k^\nu)]^2\delta_{kj}. \tag{11.3}$$

Here, δ_{kj} *is Kronecker's symbol.*

Proof. Put in Lommel's integral (9.1) $\zeta = 1$, $\alpha = \alpha_k^\nu$, and $\beta = \alpha_j^\nu$ for $k \neq j$. Since $J_\nu(\alpha_k^\nu) = J_\nu(\alpha_j^\nu) = 0$, we obtain

$$\int_0^1 tJ_\nu(\alpha_k^\nu t)J_\nu(\alpha_j^\nu t)dt = 0. \tag{11.4}$$

Putting in Lommel's integral (9.4) $\zeta = 1$ and $\alpha = \alpha_k^\nu$, we obtain

$$\int_0^1 tJ_\nu^2(\alpha_k^\nu t)dt = \frac{1}{2}[J'_\nu(\alpha_k^\nu)]^2. \tag{11.5}$$

Combining (11.4) and (11.5) and making the change of variables $t = x/l$, we obtain (11.3), as we wanted to prove.

Theorem 11.2. *Let some function* $f(x)$ *of real variable* x *can be represented by the uniformly converging functional series*

$$f(x) = \sum_{k=1}^\infty a_k J_\nu\Big(\alpha_k^\nu \frac{x}{l}\Big), \qquad \nu > -1, \tag{11.6}$$

where α_k^ν *denote the positive roots of Eq.* $J_\nu(\alpha) = 0$*, numbered in the increasing order. Then,*

$$a_k = \frac{2}{[lJ'_\nu(\alpha_k^\nu)]^2}\int_0^l xf(x)J_\nu\Big(\alpha_k^\nu \frac{x}{l}\Big)dx. \tag{11.7}$$

The series (11.6) *with the coefficients* (11.7) *is called the Fourier–Bessel series.*

Proof. Since the series (11.6) is uniformly converging, it can be integrated term by term. Multiply equality (11.6) by $xJ_\nu(\alpha_n^\nu x/l)$ and integrate the resulting equality in the limits $[0, l]$ to get

$$\int_0^l xf(x)J_\nu\Big(\alpha_n^\nu \frac{x}{l}\Big)dx = \sum_{k=1}^\infty a_k \int_0^l xJ_\nu\Big(\alpha_n^\nu \frac{x}{l}\Big)J_\nu\Big(\alpha_k^\nu \frac{x}{l}\Big)dx$$

and, using (11.3), find

$$\int_0^l xf(x)J_\nu\Big(\alpha_n^\nu \frac{x}{l}\Big)dx = \sum_{k=1}^\infty a_k \frac{l^2}{2}[J'_\nu(\alpha_k^\nu)]^2\delta_{kn} = \frac{l^2}{2}[J'_\nu(\alpha_n^\nu)]^2 a_n,$$

proving the theorem.

Theorem 11.3 (Hobson's theorem). *Let the function $f(x)$ satisfy the following conditions:*

1. *there exists the integral*

$$\int\limits_0^l t^{1/2}|f(t)|dt < \infty;$$

2. $|f(x)| < \infty$ *for* $x \in [0, l]$.

Then the series $\sum\limits_{k=1}^{\infty} a_k J_\nu(\alpha_k^\nu x/l)$ *converges for* $\nu > -1/2$, $x \in]0, l[$, *and its sum is*

$$\frac{1}{2}[f(x+0) + f(x-0)].$$

For proof see, e.g., in [21], Chap. XVIII.

♦ If γ_k^ν are positive roots of the equation

$$xJ'_\nu(x) + HJ_\nu(x) = 0, \tag{11.8}$$

where $H =$ const independent of x and ν, then the series

$$f(x) = \sum_{m=1}^{\infty} b_m J_\nu\Big(\gamma_m^\nu \frac{x}{l}\Big), \tag{11.9}$$

where

$$b_m = \frac{2(\gamma_m^\nu)^2}{[(\gamma_m^\nu)^2 - \nu^2]J_\nu^2(\gamma_m^\nu) + [\gamma_m^\nu J'_\nu(\gamma_m^\nu)]^2}\frac{1}{l^2}\int\limits_0^l xf(x)J_\nu\Big(\gamma_m^\nu \frac{x}{l}\Big)dx, \tag{11.10}$$

is called *the Dini series* of the function $f(x)$.

Theorem 11.4. *The functions* $J_\nu(\gamma_m^\nu x/l)$, $m = \overline{1,\infty}$, $\nu > -1$, *where* γ_m^ν *are roots of Eq.* (11.8), *possess, on the interval* $]0, l[$, *the property of orthogonality with weight function* $\rho(x) = x$, *i.e.,*

$$\int\limits_0^l xJ_\nu\Big(\gamma_m^\nu \frac{x}{l}\Big)J_\nu\Big(\gamma_k^\nu \frac{x}{l}\Big)dx = \frac{l^2}{2}\Big\{\Big[1 - \frac{\nu^2}{(\gamma_m^\nu)^2}\Big]J_\nu^2(\gamma_m^\nu) + [J'_\nu(\gamma_m^\nu)]^2\Big\}\delta_{km}. \tag{11.11}$$

Proof literally repeats that of relation (11.3).

Example 11.1. Expand the function $f(x) = x^0 = 1$ in the Fourier–Bessel series $\sum\limits_{k=1}^{\infty} a_k J_0(\alpha_k^0 x)$, with α_k^0 being roots of the Bessel function $J_0(x)$, on the interval $]0, 1[$.

Solution. Write

$$a_k = \frac{2\int\limits_0^1 x^0 x J_0(\alpha_k^0 x)dx}{[J'_0(\alpha_k^0)]^2}.$$

Since $J_0'(x) = -J_1(x)$, then the change of variables $t = \alpha_k^0 x$ yields

$$a_k = \frac{2\int\limits_0^{\alpha_k^0} tJ_0(t)dt}{(\alpha_k^0)^2 J_1^2(\alpha_k^0)}.$$

Using the recurrence relation

$$\frac{d}{dz}[z^{\nu+1}J_{\nu+1}(z)] = z^{\nu+1}J_\nu(z),$$

with $\nu = 0$ we find

$$a_k = \frac{2}{(\alpha_k^0)^2 J_1^2(\alpha_k^0)} tJ_1(t)\Big|_0^{\alpha_k^0} = \frac{2}{\alpha_k^0 J_1(\alpha_k^0)}.$$

Thus,

$$\sum_{k=1}^{\infty} \frac{2J_0(\alpha_k^0 x)}{\alpha_k^0 J_1(\alpha_k^0)} = 1, \quad x \in]0,1[.$$

12 The Sturm–Liouville problem for Bessel's equation

The Sturm–Liouville problem for Bessel's equation plays an important role in solving a broad class of partial differential equations. Its standard formulation is

Find the eigenvalues and eigenfunctions of the problem

$$\begin{gathered} x^2y'' + xy' + (\lambda x^2 - \nu^2)y = 0, \qquad \nu > 0, \\ \lim_{x\to 0} |y(x)| < \infty, \qquad ly'(l) + Hy(l) = 0, \quad H > 0, \quad l > 0. \end{gathered} \tag{12.1}$$

Write the orthogonality relationship for the eigenfunctions of the problem.

Solution is performed by the scheme used earlier for other ordinary differential equations.

1. Let $\lambda = \omega^2$, $\omega > 0$. The general solution of Eq. (12.1) has the form

$$y(x) = C_1 J_\nu(\omega x) + C_2 N_\nu(\omega x).$$

By definition

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k},$$

whence

$$J_\nu(0) = \begin{cases} 0 & \nu > 0, \\ 1 & \nu = 0. \end{cases}$$

Hence, $|J_\nu(0)| < \infty$ for $\nu > 0$. Since

$$\lim_{x\to 0} |N_\nu(x)| = \infty,$$

then $C_2 = 0$.

Consider the second condition

$$C_1[\omega l J_\nu'(\omega l) + HJ_\nu(\omega l)] = 0.$$

The bracketed expression is equal to zero if $\omega l = \gamma_k^\nu$, $k = \overline{1, \infty}$, where γ_k^ν are roots of the equation $\gamma J_\nu'(\gamma) + H J_\nu(\gamma) = 0$. Hence,

$$\lambda_k = \left(\frac{\gamma_k^\nu}{l}\right)^2, \qquad y_k(x) = A_k J_\nu\left(\gamma_k^\nu \frac{x}{l}\right). \tag{12.2}$$

2. Let $\lambda = -\omega^2$, $\omega > 0$. Then the general solution of the equation

$$x^2 y'' + x y' - (\omega^2 x^2 + \nu^2) y = 0$$

has the form

$$y(x) = C_1 I_\nu(\omega x) + C_2 K_\nu(\omega x).$$

Similar to the previous case,

$$|\lim_{x\to 0} I_\nu(x)| < \infty, \qquad \text{and} \qquad |\lim_{x\to 0} K_\nu(x)| = \infty.$$

Hence,

$$C_2 = 0.$$

By definition,

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu + k + 1)} \left(\frac{x}{2}\right)^{2k+\nu}.$$

Then,

$$\omega l I'(\omega l) + H I(\omega l) = \sum_{k=0}^{\infty} \frac{2k + \nu + H}{k!\Gamma(\nu + k + 1)} \left(\frac{l\omega}{2}\right)^{2k+\nu} = 0.$$

This equality is impossible since its left side is positive for $\omega l \neq 0$. Then,

$$C_1 = 0 \qquad \text{and} \qquad y(x) = 0, \tag{12.3}$$

that is, only the trivial solution exists.

3. Let $\lambda = 0$. In this case,

$$x^2 y'' + x y' - \nu^2 y = 0,$$

and it is necessary to consider two variants: $\nu = 0$ and $\nu \neq 0$.

(a) Put $\nu = 0$ and rewrite the equation as

$$x y'' + y' = \frac{d}{dx}[x y'(x)] = 0,$$

whence

$$x y' = C_1,$$

and the general solution is

$$y(x) = C_1 \ln x + C_2.$$

From the first condition we have $C_1 = 0$ and from the second one $C_2 = 0$.

(b) Put $\nu > 0$. Seek a particular solution in the form $y = x^\alpha$. Substituting this into the original equation yields

$$\alpha(\alpha - 1) x^\alpha + \alpha x^\alpha - \nu^2 x^\alpha = 0.$$

Hence, $\alpha = \pm\nu$. Then,

$$y(x) = C_1 x^\nu + C_2 x^{-\nu}.$$

From the first condition of (12.1) we find

$$C_2 = 0,$$

and from the second one

$$C_1(\nu l^\nu + Hl^\nu) = 0$$

it follows that

$$C_1 = 0.$$

Thus, the solution of the Sturm–Liouville problem (12.1) is

$$\lambda_k = \left(\frac{\gamma_k^\nu}{l}\right)^2, \qquad y_k(x) = C_1 J_\nu\left(\gamma_k^\nu \frac{x}{l}\right).$$

4. The orthogonality relationship for the functions (12.2), in view of (11.11), takes the form

$$\int\limits_0^l x y_n(x) y_k(x) dx = C_m C_k \int\limits_0^l x J_\nu\left(\gamma_m^\nu \frac{x}{l}\right) J_\nu\left(\gamma_k^\nu \frac{x}{l}\right) dx =$$
$$= C_m^2 \frac{l^2}{2}\left\{\left[1 - \frac{\nu^2}{(\gamma_m^\nu)^2}\right] J_\nu^2(\gamma_m^\nu) + [J'_\nu(\gamma_m^\nu)]^2\right\}\delta_{km}. \qquad (12.4)$$

CHAPTER 3
Classical orthogonal polynomials

13 Legendre polynomials

When studying the distribution of Newtonian potentials in a ball, Legendre introduced polynomials of a special type, later called the Legendre polynomials, and investigated their properties.

◊ Orthogonal Legendre polynomials can be defined by
(1) a generating function;
(2) a corresponding differential equation;
(3) an integral representation;
(4) an orthogonalization procedure for functions $f(x) = x^k$.

Note that Bessel functions of integer order can also be defined with the help of a generating function (see Sec. "Generating functions. The Bessel integral").

♦ The function

$$\Psi(t,x) = \frac{1}{\sqrt{1+t^2-2tx}}, \qquad |t|<1, \quad -1 \leqslant x \leqslant 1 \tag{13.1}$$

is called the generating function for the Legendre polynomials.

Putting

$$x = \cos\theta, \qquad 0 \leqslant \theta \leqslant \pi,$$

we find

$$1+t^2-2tx = 1+t^2-2t\cos\theta = (t-e^{i\theta})(t-e^{-i\theta}).$$

Thus, $\Psi(t,x)$, as a function of variable t on the interval $-1 \leqslant x \leqslant 1$, has two singular points at $t = t_{1,2} = \exp(\pm i\theta)$. Both points lie on the unit circle $|t| = 1$. Hence, $\Psi(t,x)$, as a function of t, is analytic inside the unit circle at any fixed real x, $|x| \leqslant 1$ and can be expanded in this circle in a converging Taylor power series in t.

Theorem 13.1. *The coefficients of the expansion of the function* $\Psi(t,x)$ (13.1) *in the Taylor power series in* t

$$\Psi(t,x) = \sum_{n=0}^{\infty} P_n(x)t^n \tag{13.2}$$

are polynomials of degree n *in* x.

The polynomials $P_n(x)$ *are called the Legendre polynomials of degree* n.

Proof. Obviously, we have

$$P_n(x) = \frac{1}{n!}\frac{\partial^n\Psi}{\partial t^n}\Big|_{t=0}.$$

On the other hand, the nth derivative of an analytic function is given by the Cauchy integral formula

$$\frac{n!}{2\pi i}\oint_\gamma \frac{\Psi(\omega,x)}{\omega^{n+1}}d\omega = \frac{\partial^n\Psi(t,x)}{\partial t^n}\Big|_{t=0}, \tag{13.3}$$

where γ is an arbitrary piecewise smooth closed path in the complex plane of ω, which encloses the point $\omega = 0$ and lies inside the unit circle.

Make the change of variables

$$\sqrt{1-2x\omega+\omega^2} = 1-\omega z, \tag{13.4}$$

in the integral (13.3) to get

$$1 - 2x\omega + \omega^2 = 1 - 2\omega z + \omega^2 z^2,$$
$$2(z - x) = (z^2 - 1)\omega, \qquad \omega = \frac{2(x - z)}{1 - z^2}. \tag{13.5}$$

Upon this transformation, the point $\omega = 0$ maps into the point $z = x$. Then

$$d\omega = d\Big(\frac{2(z - x)}{z^2 - 1}\Big) = 2\Big[\frac{dz}{z^2 - 1} - \frac{z - x}{(z^2 - 1)^2}2zdz\Big] =$$
$$= \frac{2(z^2 - 1 - 2z^2 + 2zx)}{(z^2 - 1)^2}dz = -2\frac{1 + z^2 - 2xz}{(z^2 - 1)^2}dz;$$

Similarly, in view of (13.5), we obtain

$$1 - \omega z = \frac{(1 - z^2) - 2z(x - z)}{1 - z^2} = \frac{1 + z^2 - 2zx}{1 - z^2}.$$

Thus,

$$\Psi(\omega, x)d\omega = \frac{d\omega}{1 - \omega z} = -\frac{z^2 - 1}{1 + z^2 - 2zx}\frac{(-2)(1 + z^2 - 2zx)}{(z^2 - 1)^2}dz = \frac{2dz}{z^2 - 1}.$$

Hence,

$$\oint\limits_{\gamma} \frac{\Psi(\omega, x)}{\omega^{n+1}}d\omega = \oint\limits_{\tilde\gamma} \Big[\frac{z^2 - 1}{2(z - x)}\Big]^{n+1}\frac{2dz}{z^2 - 1} = \frac{1}{2^n}\oint\limits_{\tilde\gamma} \frac{(z^2 - 1)^n}{(z - x)^{n+1}}dz,$$

where $\tilde\gamma$ is a piecewise smooth closed path enclosing the point $z = x$. (In the last integral, the integrand, as a function of z, is analytic everywhere except for the point $z = x$. Hence, the closed path $\tilde\gamma$ is now arbitrary if it encloses the point $z = x$.) Thus,

$$P_n(x) = \frac{1}{2\pi i}\frac{1}{2^n}\oint\limits_{\tilde\gamma} \frac{(z^2 - 1)^n}{(z - x)^{n+1}}dz. \tag{13.6}$$

Consider

$$\frac{d^n}{dx^n}\int\limits_{\tilde\gamma} \frac{(z^2 - 1)^n}{z - x}dz = n!\oint\limits_{\tilde\gamma} \frac{(z^2 - 1)^n}{(z - x)^{n+1}}dz$$

and note that

$$\frac{1}{2\pi i}\oint\limits_{\tilde\gamma} \frac{(z^2 - 1)^n}{z - x}dz = (x^2 - 1)^n.$$

From here

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n. \tag{13.7}$$

Hence, $P_n(x)$ is a polynomial of degree n, Q. E. D.

♦ Formulas (13.7) are called Rodrigues's formulas and relation (13.6) is called the Schläffly integral.

Corollary. The relation

$$P_n(-x) = (-1)^n P_n(x)$$

is valid.

Proof. By definition,

$$\frac{1}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} P_n(x) z^n.$$

Replace x by $-x$ and z by $-z$:

$$\frac{1}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} P_n(-x)(-z)^n.$$

By comparing the two series, we obtain $P_n(-x) = (-1)^n P_n(x)$.

Example 13.1. Show that the relations

$$P_n(1) = 1, \qquad P_n(-1) = (-1)^n$$

are valid.

Solution. Actually, putting $x = 1$ and $x = -1$ in (13.1), we obtain

$$\Psi(t,1) = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} P_n(1) t^n,$$
$$\Psi(t,-1) = \frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n = \sum_{n=0}^{\infty} P_n(-1) t^n.$$

The first equation yields $P_n(1) = 1$ and the second one $P_n(-1) = (-1)^n$.

Example 13.2. Find the value of $P_n(0)$ for any n.

Solution. Similar to Example 13.1, put $x = 0$ in relation (13.1) to get

$$\Psi(t,0) = \frac{1}{\sqrt{1+t^2}} = \sum_{n=0}^{\infty} P_n(0) t^n.$$

On the other hand,

$$\frac{1}{\sqrt{1+t^2}} = \sum_{k=0}^{\infty} (-1)^k \frac{(2k-1)!!}{(2k)!!} t^{2k}.$$

From here

$$P_{2k+1}(0) = 0, \qquad P_{2k}(0) = (-1)^k \frac{(2k-1)!!}{(2k)!!}, \quad k = \overline{0,\infty}.$$

Example 13.3. Find the explicit form of the first six Legendre polynomials.

Solution. Putting $n = \overline{0,5}$ in (13.7), we obtain the functions

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x),$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

whose plots are given in Figs. 10 and 11.

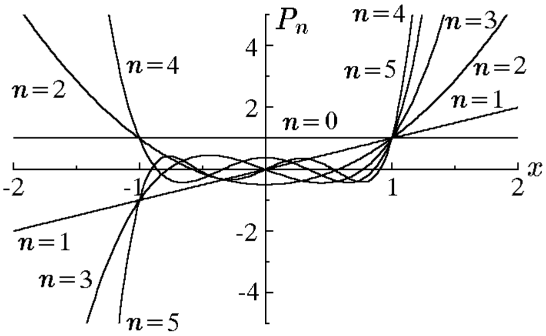

Fig. 10.

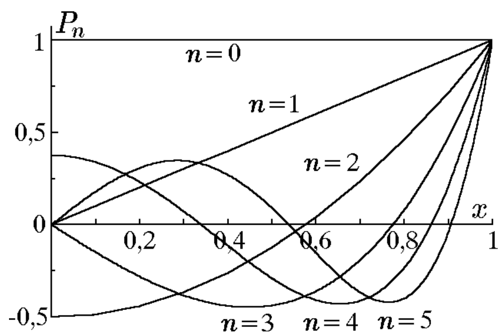

Fig. 11.

Example 13.4. Show that for Legendre polynomials the representation

$$P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}$$

is valid.

Solution. According to Rodrigues's formula,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n.$$

Since

$$(x^2-1)^n = x^{2n} - nx^{2n-2} + \frac{n(n-1)}{2!} x^{2n-4} - \frac{n(n-1)(n-2)}{2!} x^{2n-6} + \ldots + (-1)^n,$$

that is,

$$(x^2-1)^n = \sum_{k=0}^{n} (-1)^k C_n^k x^{2n-2k},$$

then,

$$\frac{d^n}{dx^n} (x^2-1)^n = \sum_{k=0}^{[n/2]} (-1)^k C_n^k (2n-2k)(2n-2k-1)\cdots(n+1-2k) x^{n-2k}$$

or

$$\frac{d^n}{dx^n} (x^2-1)^n = \sum_{k=0}^{[n/2]} (-1)^k \frac{n!(2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k}.$$

Thus,

$$P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}.$$

14 Recurrence relations for Legendre polynomials

Statement 14.1. *The recurrence relations*

$$(n+1)P_{n+1}(x) - x(2n+1)P_n(x) + nP_{n-1}(x) = 0; \tag{14.1}$$
$$nP_n(x) - xP_n'(x) + P_{n-1}'(x) = 0 \tag{14.2}$$

are valid.

1. Differentiate the generating function $\Psi(t,x)$ (8.1) with respect to x to get

$$\frac{\partial\Psi(t,x)}{\partial x} = \frac{t}{(1+t^2-2tx)^{3/2}}.$$

Hence,

$$(1+t^2-2tx)\frac{\partial\Psi}{\partial x} - t\Psi = 0. \tag{14.3}$$

Similarly, differentiate this function with respect to t to get

$$\frac{\partial\Psi(t,x)}{\partial t} = \frac{2t-2x}{(-2)(1+t^2-2tx)^{3/2}},$$

which yields

$$(1+t^2-2tx)\frac{\partial\Psi}{\partial t} + (t-x)\Psi = 0. \tag{14.4}$$

Multiply (14.3) by $(t-x)$ and (14.4) by t and combine the results to get

$$(t-x)\frac{\partial\Psi}{\partial x} + t\frac{\partial\Psi}{\partial t} = 0. \tag{14.5}$$

Substitute

$$\Psi(t,x) = \sum_{k=0}^{\infty} P_k(x)t^k \quad \text{and} \quad \frac{\partial\Psi(t,x)}{\partial t} = \sum_{k=0}^{\infty} kP_k(x)t^{k-1}$$

into (14.4). Then,

$$(1+t^2-2tx)\sum_{k=1}^{\infty} kP_k(x)t^{k-1} + (t-x)\sum_{k=0}^{\infty} P_k(x)t^k = 0.$$

Hence,

$$\underbrace{\sum_{k=0}^{\infty} kP_k(x)t^{k-1}}_{k-1=n} + \underbrace{\sum_{k=0}^{\infty} kP_k(x)t^{k+1}}_{k+1=n} - \underbrace{\sum_{k=0}^{\infty} 2kxP_k(x)t^k}_{k=n} + \underbrace{\sum_{k=0}^{\infty} P_k(x)t^{k+1}}_{k+1=n} - \underbrace{\sum_{k=0}^{\infty} xP_k(x)t^k}_{k=n} = 0.$$

Change the summation index in the resulting sums by the rules given beneath the parentheses:

$$\sum_{n=1}^{\infty}(n+1)P_{n+1}(x)t^n + P_1(x) + \sum_{n=1}^{\infty}(n-1)P_{n-1}(x)t^n -$$
$$-\sum_{n=1}^{\infty} 2nxP_n(x)t^n + \sum_{n=1}^{\infty} P_{n-1}(x)t^n - \sum_{n=1}^{\infty} xP_n(x)t^n - xP_0(x) = 0.$$

In view of the relation $P_1(x) = xP_0(x)$, we obtain

$$\sum_{n=1}^{\infty}[(n+1)P_{n+1}(x) + (n-1)P_{n-1}(x) - 2nxP_n(x) + P_{n-1}(x) - xP_n(x)]t^n = 0,$$

Equating the coefficients of terms with identical powers of t to zero, we arrive at (14.1).

Perform similar manipulations with relation (14.5):

$$\sum_{n=1}^{\infty} nP_n(x)t^n + \sum_{n=0}^{\infty} P_n'(x)t^{n+1} - \sum_{n=0}^{\infty} xP_n'(x)t^n = 0.$$

Put $n+1 = n$ in the second sum. Then n varies within the limits $n = \overline{1,\infty}$. In view of $P_0'(x) = 0$, we obtain

$$\sum_{n=1}^{\infty}[nP_n - xP_n' + P_{n-1}']t^n = 0.$$

Thus, the statement is valid.

Statement 14.2. *The relations*

$$(n+1)P_n(x) - P_{n+1}'(x) + xP_n'(x) = 0, \tag{14.6}$$

$$P_{n+1}'(x) - P_{n-1}'(x) - (2n+1)P_n(x) = 0 \tag{14.7}$$

are valid.

Actually, differentiating (14.1) with respect to x, we have

$$(n+1)P_{n+1}'(x) - (2n+1)P_n(x) - (2n+1)xP_n'(x) + nP_{n-1}'(x) = 0.$$

Eliminating P_{n-1}' with the help of (14.2), we obtain

$$(n+1)P_{n+1}'(x) - (2n+1)P_n(x) - (2n+1)xP_n'(x) + n(xP_n'(x) - nP_n(x)) = 0$$

or

$$(n+1)P_{n+1}'(x) - (n+1)xP_n'(x) - (n+1)^2P_n(x) = 0.$$

Thus, relation (14.6) is proved.

Combining (14.2) with (14.6), we obtain (14.7).

Statement 14.3. *The relation*

$$\int P_n(x)dx = \frac{1}{2n+1}[P_{n+1}(x) - P_{n-1}(x)] + C, \tag{14.8}$$

where C is an arbitrary constant is valid.

Relation (14.8) immediately follows from Eq. (14.7).

Statement 14.4. *The Legendre polynomials $y = P_n(x)$ satisfy the equation*

$$[(1-x^2)y']' + n(n+1)y = 0. \tag{14.9}$$

Actually, let $U(x) = (x^2-1)^n$. Then,

$$U'(x) = n2x(x^2-1)^{n-1}$$

or

$$(x^2-1)U'(x) - 2nxU(x) = 0.$$

Differentiate this relation $n+1$ times with respect to x:

$$[(x^2-1)U'(x)]^{(n+1)} - 2n[xU(x)]^{(n+1)} = 0. \tag{14.10}$$

By Leibniz's formula

$$(UV)^{(n)} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} U^{(n-k)} V^{(k)},$$

we have

$$[(x^2-1)U'(x)]^{(n+1)} = (x^2-1)U^{(n+2)}(x) + (n+1)2xU^{(n+1)}(x) + \frac{(n+1)n}{2} 2U^{(n)}(x);$$
$$2n[xU(x)]^{(n+1)} = xU^{(n+1)}(x) + (n+1)U^{(n)}(x).$$

Substituting this into the relation (14.10) yields

$$(x^2-1)[U^{(n)}(x)]'' + 2x(n+1)[U^{(n)}(x)]' + n(n+1)U^{(n)}(x) -$$
$$-2n\{x[U^{(n)}(x)]' + (n+1)U^{(n)}(x)\} = 0.$$

Denoting $W = U^{(n)}(x)$, we obtain

$$(x^2-1)W'' - 2x(n+1)W' + n(n+1)W = 0.$$

This is a linear differential equation, and hence $y(x) = CW(x)$, where $C = \text{const}$, is its solution as well. Put $C = 1/(2^n n!)$ to obtain

$$y = \frac{1}{n!2^n}\big[(x^2-1)^n\big]^{(n)} = P_n(x),$$

Q. E. D.

Statement 14.5. *The recurrence relation*

$$(1-x^2)P_n'(x) = \frac{n(n+1)}{2n+1}[P_{n-1}(x) - P_{n+1}(x)] \tag{14.11}$$

is valid.

Make use of the identity

$$\frac{d}{dx}[(1-x^2)P_n'(x)] = -n(n+1)P_n(x).$$

From here

$$(1-x^2)P_n'(x) = -n(n+1)\int P_n(x)dx + C$$

or

$$(1-x^2)P_n'(x) = n(n+1)\frac{1}{2n+1}[P_{n-1}(x) - P_{n+1}(x)] + C.$$

Putting $x = 1$, find $C = 0$, and this completes the proof.

Statement 14.6. *All zeros of the Legendre polynomials are simple and are localized in the interval* $]-1,1[$.

This statement is proved in the book [3], in the section devoted to the general properties of orthogonal polynomials.

Example 14.1. Prove that

$$P_n'(1) = \frac{1}{2}n(n+1).$$

Solution. We have

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0.$$

Put $x = 1$ to get

$$-2P_n'(1) + n(n+1)P_n(1) = 0.$$

However, $P_n(1) = 1$ for any n. Hence,

$$P_n'(1) = \frac{1}{2}n(n+1).$$

15 Orthogonality of Legendre polynomials

Statement 15.1. *The Legendre polynomials of different degrees are orthogonal on the interval* $[-1, 1]$ *with weight function* $\rho(x) = 1$*, that is,*

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0, \qquad n \neq m. \tag{15.1}$$

Actually,

$$\frac{d}{dx}[(1-x^2)P_n'(x)] + n(n+1)P_n(x) = 0,$$
$$\frac{d}{dx}[(1-x^2)P_m'(x)] + m(m+1)P_m(x) = 0.$$

Multiplying the first equation by $P_m(x)$ and the second by $P_n(x)$ and subtracting the second from the first we obtain

$$P_m(x)\frac{d}{dx}[(1-x^2)P_n'(x)] - P_n(x)\frac{d}{dx}[(1-x^2)P_m'(x)] +$$
$$+[(n+1)n - (m+1)m]P_n(x)P_m(x) = 0$$

or

$$\frac{d}{dx}\{(1-x^2)[P_m(x)P_n'(x) - P_n(x)P_m'(x)]\} - (n-m)(m+n+1)P_n(x)P_m(x) = 0.$$

Integrate the both sides of this equality on the interval $]-1, 1[$. Since

$$\int_{-1}^{1} \frac{d}{dx}\{(1-x^2)[P_m(x)P_n'(x) - P_n(x)P_m'(x)]\}dx =$$
$$= \{(1-x^2)[P_m(x)P_n'(x) - P_n(x)P_m'(x)]\}\Big|_{-1}^{1} = 0,$$

then

$$(m+n+1)(n-m)\int_{-1}^{1} P_n(x)P_m(x)dx = 0.$$

This implies (15.1).

Statement 15.2. *For the norm of a Legendre polynomial, the relation*

$$\|P_n\|^2 = \int_{-1}^{1} [P_n(x)]^2 dx = \frac{2}{2n+1} \tag{15.2}$$

is valid.

By Rodrigues's formula

$$J = \int_{-1}^{1} [P_n(x)]^2 dx = \frac{1}{2^{2n}(n!)^2} \int_{-1}^{1} \frac{d^n(x^2-1)}{dx^n} \frac{d^n(x^2-1)}{dx^n} dx.$$

Integrate this expression by parts n times and take into account the fact that the terms outside the integral that contain $(x^2-1)|_{-1}^{1}$ are equal to zero. Then,

$$J = \frac{(-1)^n}{2^{2n}(n!)^2} \int_{-1}^{1} (x^2-1)^n \Big[\frac{d^{2n}}{dx^{2n}}(x^2-1)^n\Big] dx.$$

Noting that

$$\frac{d^{2n}}{dx^{2n}}(x^2-1)^n = (2n)!,$$

we obtain

$$J = \frac{(-1)^n (2n)!}{2^{2n}(n!)^2} \int_{-1}^{1} (x^2-1)^n dx.$$

Making the change of variables $1 - x = 2t$, $dx = -2dt$ in the integral

$$I = \int_{-1}^{1} (x^2-1)^n dx = (-1)^n \int_{-1}^{1} (1-x^2)^n dx,$$

we find

$$I = (-1)^n 2^{2n+1} \int_{0}^{1} t^n (1-t)^n dt = (-1)^n 2^{2n+1} B(n+1, n+1) =$$

$$= (-1)^n 2^{2n+1} \frac{\Gamma(n+1)\Gamma(n+1)}{\Gamma(2n+2)} = (-1)^n \frac{(n!)^2 2^{2n+1}}{(2n+1)!}.$$

Hence,

$$J = \frac{2}{2n+1},$$

which proves the statement.

Theorem 15.1. *The Legendre polynomials $P_n(x)$ are a solution of the Sturm–Liouville problem.*

$$\frac{d}{dx}\Big[(1-x^2)\frac{dy}{dx}\Big]+\lambda y=0, \qquad (15.3)$$
$$|\lim_{|x|\to 1} y(x)|<\infty, \qquad x\in[-1,1],$$

with eigenvalues $\lambda=n(n+1)$.

♦ Equation (15.3) with any λ is called *Legendre's equation* and its solutions different from the identical zero are called *the Legendre functions*.

Example 15.1. Solve the Sturm–Liouville problem

$$y''+\operatorname{ctg} x y'+\lambda y=0, \qquad |y(0)|<\infty, \quad |y(\pi)|<\infty. \qquad (15.4)$$

Orthonormalize the eigenfunctions of the problem.

Solution. Perform the change of variables

$$t=\cos x. \qquad (15.5)$$

Since $dt/dx=-\sin x$ and

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=-\sin x\frac{dt}{dx},$$
$$\frac{d^2y}{dx^2}=\frac{d}{dx}\Big(\frac{dy}{dx}\Big)=\frac{d}{dx}\Big(-\sin x\frac{dy}{dt}\Big)=-\cos x\frac{dy}{dt}-\sin x\frac{d^2y}{dt^2}\frac{dt}{dx}=\sin^2 x\frac{d^2y}{dt^2}-\cos x\frac{dy}{dt},$$

we obtain

$$\sin^2 x\frac{d^2y}{dt^2}-\cos x\frac{dy}{dt}+\frac{\cos x}{\sin x}\Big(-\sin x\frac{dy}{dt}\Big)+\lambda y=0 \qquad (15.6)$$

or

$$(1-\cos^2 x)\frac{d^2y}{dt^2}-2\cos x\frac{dy}{dt}+\lambda y=0.$$

Eliminate x from Eq. (15.6) with the help of (15.5) in view of the fact that $t|_{x=0}=1$ and $t|_{x=\pi}=-1$. Then, to find $y(x)$, we have the following Sturm–Liouville problem:

$$(1-t^2)\frac{d^2y}{dx^2}-2t\frac{dy}{dt}+\lambda y=0, \qquad (15.7)$$
$$|y(1)|<\infty, \qquad |y(-1)|<\infty.$$

Solutions of the problem (15.7) are (see Theorem 15.1) the eigenvalues $\lambda_n=n(n+1)$ and the eigenfunctions $y_n(t)=C_nP_n(t)$ with the orthogonality relationship

$$\langle y_n(t)|y_kt)\rangle=\int\limits_{-1}^{1}C_nC_kP_n(t)P_k(t)dt=\frac{2C_n^2}{2n+1}\delta_{nk},$$

where $n,k=\overline{0,\infty}$.

Return to the original variable. Put

$$C_n = \sqrt{\frac{2n+1}{2}}$$

to obtain a solution to the problem (15.4):

$$y_n(x) = \sqrt{\frac{2n+1}{2}} P_n(\cos x), \qquad \lambda_n = n(n+1) \tag{15.8}$$

with the orthogonality relationship

$$\langle y_n(x)|y_k(x)\rangle = \sqrt{\frac{(2n+1)(2k+1)}{4}} \int\limits_0^\pi P_n(\cos x) P_k(\cos x) \sin x \, dx = \delta_{nk}$$

(see Theorem 15.1).

Example 15.2. Calculate the integral

$$I = \int\limits_0^5 x^2 P_n^2(x/5) dx. \tag{15.9}$$

Solution. Make the change of variables $x/5 = t$ in the integral. Then,

$$I = \int\limits_0^1 (5t)^2 P_n^2(t) 5dx = 125 \int\limits_0^1 [tP_n(t)]^2 dt. \tag{15.10}$$

In virtue of the evenness of the integrand, (15.10) can be rewritten as

$$I = \frac{125}{2} \int\limits_{-1}^1 [tP_n(t)]^2 dt.$$

Use the recurrence relation (14.1)

$$(n+1)P_{n+1}(t) - (2n+1)tP_n(t) + nP_{n-1}(t) = 0$$

from which

$$tP_n(t) = \frac{1}{2n+1}\big[(n+1)P_{n+1}(t) + nP_{n-1}(t)\big].$$

Hence,

$$I = \frac{125}{2} \int\limits_{-1}^1 \frac{1}{(2n+1)^2}\big[(n+1)P_{n+1}(t) + nP_{n-1}(t)\big]^2 dt =$$

$$= \frac{125}{2(2n+1)^2} \int\limits_{-1}^1 \big[(n+1)^2 P_{n+1}^2(t) + 2n(n+1)P_{n+1}(t)P_{n-1}(t) + n^2 P_{n-1}^2(t)\big] dt =$$

$$= \frac{125}{2(2n+1)^2}\Big[(n+1)^2 \int\limits_{-1}^1 P_{n+1}^2(t)dt + 2n(n+1) \int\limits_{-1}^1 P_{n+1}(t)P_{n-1}(t)dt + n^2 \int\limits_{-1}^1 P_{n-1}^2(t)dt\Big].$$

In view of the orthogonality relationship

$$\int_{-1}^{1} P_n(t)P_k(t)dt = \|P_n\|^2\delta_{nk} = \frac{2}{2n+1}\delta_{nk},$$

we obtain

$$I = \frac{125}{2(2n+1)^2}\{(n+1)^2\|P_{n+1}\|^2 + n^2\|P_{n-1}\|^2\}$$

or

$$I = \frac{125}{(2n+1)^2}\Big\{\frac{(n+1)^2}{2n+3} + \frac{n^2}{2n-1}\Big\}.$$

16 The Fourier–Legendre series

◊ *It can be shown that any function $f(x)$ satisfying the Dirichlet conditions on the interval $[-1, 1]$ is expandable in a series in Legendre polynomials.*

$$f(x) = \sum_{k=0}^{\infty} a_k P_k(x), \qquad a_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)P_k(x)dx. \tag{16.1}$$

The series (16.1) *converges to the function $f(x)$ at each point where this function is continuous and to the function $f^*(x) = \frac{1}{2}(f(x+0)+f(x-0))$ at points of discontinuity.*

♦ The series (16.1) is called *the Fourier–legendre series.*

◊ Recall that a function $f(x)$ satisfies the Dirichlet conditions on the interval $[a, b]$ if

(1) it has a finite number of extremums on $[a, b]$;

(2) it is continuous on $[a, b]$ except, perhaps, for a finite number of discontinuity points of the first kind.

◊ In the case of an infinite interval ($a = -\infty$ and/or $b = \infty$), a function $f(x)$ satisfies the Dirichlet conditions if it obeys, besides conditions 1 and 2 which hold on any finite interval, the additional condition of absolute (or square) integrability

$$\int_{a}^{b} |f(x)|dx < \infty.$$

Example 16.1. Find the coefficients of the Fourier series in Legendre polynomials for the function $f(x) = |x|$.

Solution. We have

$$|x| = \sum_{n=0}^{\infty} C_n P_n(x), \qquad -1 \leqslant x \leqslant 1,$$

where

$$C_n = \Big(n + \frac{1}{2}\Big)\int_{-1}^{1} |x|P_n(x)dx.$$

Obviously, $C_{2k+1} = 0$, $k = \overline{1, \infty}$. Calculate

$$C_{2k} = \frac{4k+1}{2} \int\limits_{-1}^{1} |x| P_k(x) dx = (4k+1) \int\limits_{0}^{1} x P_{2k}(x) dx, \quad k = \overline{1, \infty}.$$

For $k = 0$ we readily find $C_0 = 1/2$. For $k = \overline{1, \infty}$ we perform integration by parts, putting $U = x$, $dV = P_{2k}(x)dx$. Then

$$dU = dx, \qquad V = \frac{1}{4k+1}[P_{2k+1}(x) - P_{2k-1}(x)],$$

and we obtain

$$C_{2k} = x[P_{2k+1}(x) - P_{2k-1}(x)]\Big|_0^1 - \int\limits_0^1 [P_{2k+1}(x) - P_{2k-1}(x)]dx = \tag{16.2}$$

$$= -\frac{P_{2k+2}(x) - P_{2k}(x)}{4k+3}\Big|_0^1 + \frac{P_{2k}(x) - P_{2k-2}(x)}{4k-1}\Big|_0^1 = \tag{16.3}$$

$$= \frac{P_{2k+2}(0) - P_{2k}(0)}{4k+3} - \frac{P_{2k}(0) - P_{2k-2}(0)}{4k-1}. \tag{16.4}$$

Transform the obtained expression. From the recurrence relation

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

for $x = 0$ we find

$$(n+1)P_{n+1}(0) = -nP_{n-1}(0).$$

Then,

$$P_{2k+2}(0) = -\frac{2k+1}{2k+2}P_{2k}(0), \qquad P_{2k-2}(0) = -\frac{2k}{2k-1}P_{2k}(0).$$

Substituting these expressions into (16.4) yields

$$C_{2k} = \Big[\frac{1}{4k-1}\Big(-\frac{2k}{2k-1} - 1\Big) + \frac{1}{4k+3}\Big(-\frac{2k+1}{2k+2} - 1\Big)\Big]P_{2k}(0)$$

or

$$C_{2k} = -\frac{4k+1}{(2k-1)(2k+2)}P_{2k}(0).$$

Thus,

$$C_{2k} = (-1)^{k+1}\frac{(4k+1)(2k-1)!!}{(2k-1)(2k+2)(2k)!!}, \quad k = \overline{1, \infty}$$

or

$$C_{2k} = (-1)^{k+1}\frac{(4k+1)(2k-2)!}{2^{2k}(k-1)!(k+1)!}, \quad k = \overline{1, \infty}.$$

Example 16.2. Expand the function

$$f(x) = \begin{cases} 0 & \text{for} \quad -1 \leqslant x < \alpha; \\ 1 & \text{for} \quad \alpha < x \leqslant 1 \end{cases}$$

in a Fourier–Legendre series.

Solution. The function satisfies the expandability conditions. Then,

$$f(x) = \sum_{n=0}^{\infty} C_n P_n(x),$$

where

$$C_n = \Big(n+\frac{1}{2}\Big)\int\limits_{-1}^{1} f(x)P_n(x)dx, \quad n=\overline{0,\infty}.$$

We have

$$C_n = \Big(n+\frac{1}{2}\Big)\int\limits_{\alpha}^{1} P_n(x)dx = \Big(n+\frac{1}{2}\Big)\frac{1}{2n+1}[P_{n+1}(x)-P_{n-1}(x)]\Big|_{\alpha}^{1} =$$

$$= -\frac{1}{2}[P_{n+1}(\alpha)-P_{n-1}(\alpha)], \quad n=\overline{1,\infty}$$

and

$$C_0 = \frac{1}{2}\int\limits_{\alpha}^{1} P_0(x)dx = \frac{1}{2}(1-\alpha).$$

Here, we have made use of the fact that $P_0(x)=1$. The sought-for expansion is written in the form

$$f(x) = \frac{1}{2}(1-\alpha) - \frac{1}{2}\sum_{n=1}^{\infty}[P_{n+1}(\alpha)-P_{n-1}(\alpha)]P_n(x).$$

In particular, if

$$f(x) = \begin{cases} 0 & \text{for } -1 \leqslant x < 0, \\ 1 & \text{for } 0 < x \leqslant 1, \end{cases}$$

we obtain

$$f(x) = \frac{1}{2} - \frac{1}{2}\sum_{n=1}^{\infty}[P_{n+1}(0)-P_{n-1}(0)]P_n(x),$$

but

$$P_{n+1}(0)-P_{n-1}(0) = \begin{cases} 0, & n=2k,\ k=\overline{1,\infty}; \\ (-1)^{k+1}\dfrac{(2k)!(4k+3)}{2^{2k+1}k!(k+1)!}, & n=2k+1,\ k=\overline{0,\infty}. \end{cases}$$

Then,

$$f(x) = \frac{1}{2} + \sum_{k=0}^{\infty}(-1)^k\frac{(4k+3)(2k)!}{2^{2k+2}k!(k+1)!}P_{2k+1}(x).$$

Example 16.3. Expand the function $f(x)=x^2-x+1$ in a Fourier series in Legendre polynomials on the interval $]-1,1[$.

Solution. Since $f(x)$ is a polynomial of the second degree, then in the expansion

$$f(x) = \sum_{n=0}^{\infty} C_n P_n(x)$$

the coefficients $C_n = 0$ for $n > 2$, that is,

$$f(x) = C_0 P_0(x) + C_1 P_1(x) + C_2 P_2(x).$$

The coefficients C_0, C_1, and C_2 can be calculated by the formula

$$C_n = \Big(n + \frac{1}{2}\Big) \int\limits_{-1}^{1} f(x) P_n(x) dx$$

for $n = 0, 1$, and 2 or directly, using the explicit form of the Legendre polynomials

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{1}{2}(3x^2 - 1),$$

that is,

$$x^2 - x + 1 = C_0 \cdot 1 + C_1 x + C_2 \frac{1}{2}(3x^2 - 1) = \frac{3}{2} C_2 x^2 + C_1 x + C_0 - \frac{C_2}{2}.$$

Finally, we obtain

$$x^2 - x + 1 = \frac{4}{3} P_0(x) - P_1(x) + \frac{2}{3} P_2(x).$$

17 Associated Legendre functions

♦ The functions

$$P_n^m(x) = (1 - x^2)^{m/2} P_n^{(m)}(x) \tag{17.1}$$

are called *the associated Legendre functions* (see Figs. 12 and 13).

◊ Note that for $m > n$ the associated Legendre functions $P_n^m(x)$ are identically equal to zero and for $m = 0$ they coincide with the Legendre polynomials, i.e., $P_n^0 = P_n(x)$.

Example 17.1. Find the explicit form of the Legendre functions $P_1^1(x)$, $P_2^1(x)$.

Solution. Actually, we have $P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$. Hence, $P_1^{(1)}(x) = 1$, $P_2^{(1)}(x) = 3x$. Finally, we obtain

$$P_1^1(x) = \sqrt{1 - x^2}, \qquad P_2^1(x) = 3x\sqrt{1 - x^2}.$$

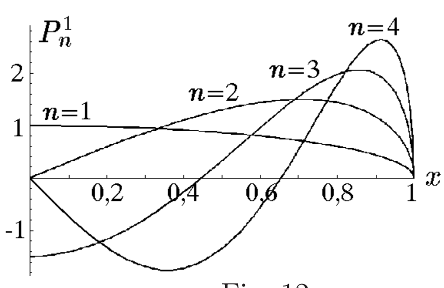

Fig. 12.

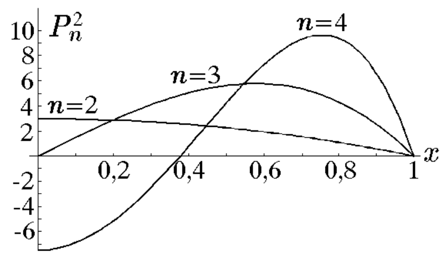

Fig. 13.

Theorem 17.1. *The associated Legendre functions P_n^m satisfy the equation*

$$\frac{d}{dx}\Big[(1-x^2)\frac{dy}{dx}\Big] + \Big[n(n+1) - \frac{m^2}{1-x^2}\Big]y = 0. \tag{17.2}$$

Equation (17.2) *is called Legendre's equation of order m or an equation for the associated Legendre functions.*

Proof. Differentiate the Legendre equation

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0$$

m times with respect to x and make use of the Leibniz formula. Then

$$\begin{gathered}(1-x^2)P_n^{(m+2)} - 2xmP_n^{(m+1)} - \frac{2m(m-1)}{2}P_n^{(m)} - \\ -2xP_n^{(m+1)} - 2mP_n^{(m)} + (n+1)nP_n^{(m)} = 0\end{gathered}$$

or

$$(1-x^2)[P_n^{(m)}]'' - 2x(m+1)[P_n^{(m)}]' - [m(m+1) - n(n+1)]P_n^{(m)} = 0. \tag{17.3}$$

Let $y(x) = (\sqrt{1-x^2})^m P_n^{(m)}$. Then

$$\begin{aligned}&P_n^{(m)}(x) = (1-x^2)^{-m/2}y(x),\\ &[P_n^{(m)}(x)]' = mx(1-x^2)^{-(m+2)/2}y(x) + (1-x^2)^{-m/2}y'(x),\\ &[P_n^{(m)}(x)]'' = m(1-x^2)^{-(m+2)/2}y(x) + m(m+2)x^2(1-x^2)^{-(m+4)/2}y(x) + \\ &\qquad +2mx(1-x^2)^{-(m+2)/2}y'(x) + (1-x^2)^{-m/2}y''(x).\end{aligned}$$

Substituting these expressions into formula (17.3) we obtain

$$\begin{gathered}(1-x^2)[m(1-x^2)^{-(m+2)/2}y(x) + m(m+2)x^2(1-x^2)^{-(m+4)/2}y(x) + \\ +2mx(1-x^2)^{-(m+2)/2}y'(x) + (1-x^2)^{-m/2}y''(x)] - \\ -2x(m+1)[mx(1-x^2)^{-(m+2)/2}y(x) + (1-x^2)^{-m/2}y'(x)] - \\ -[m(m+1) - n(n+1)](1-x^2)^{-m/2}y(x) = 0.\end{gathered}$$

Reduce this relation by the multiplier $(1-x^2)^{-m/2}$ and collect similar terms to obtain

$$(1-x^2)y'' - 2xy' + n(n+1)y - \frac{m^2}{1-x^2}y = 0,$$

proving the theorem.

Corollary. For the associated Legendre functions, the following recurrence relations are valid:

$$P_n^{m+1}(x) - \frac{2mx}{\sqrt{1-x^2}}P_n^m(x) + (n+m)(n-m+1)P_n^{m-1}(x) = 0. \tag{17.4}$$

$$[(1-x^2)^{m/2}P_n^m(x)]' + (n+m)(n-m+1)(1-x^2)^{(m-1)/2}P_n^{m-1}(x). \tag{17.5}$$

Proof. Changing m by $m-1$ in relation (17.3), we have

$$(1-x^2)P_n^{(m+1)} - 2xmP_n^{(m)} - [n(n+1)-m(m-1)]P_n^{(m-1)} = 0.$$

In view of the equality

$$n(n+1)-m(m-1) = n^2 - m^2 + n + m = (n+m)(n-m+1),$$

we can write

$$(1-x^2)P_n^{(m+1)} - 2xmP_n^{(m)} + (n+m)(n-m+1)P_n^{(m-1)} = 0. \tag{17.6}$$

Multiplying (17.6) by $(1-x^2)^{(m-1)/2}$, we obtain the equation

$$(1-x^2)^{(m+1)/2}P_n^{(m+1)}(x) - \frac{2mx}{\sqrt{1-x^2}}(1-x^2)^{m/2}P_n^{(m)}(x) +$$
$$+(n+m)(n-m+1)(1-x^2)^{(m-1)/2}P_n^{(m-1)}(x) = 0,$$

which, in view of the definition (17.1), implies the recurrence relation (17.4).

Multiplying (17.6) by $(1-x^2)^{m-1}$, we find

$$(1-x^2)^m P_n^{(m+1)} - 2mx(1-x^2)^{m-1}P_n^{(m)} +$$
$$+(n+m)(n-m+1)(1-x^2)^{m-1}P_n^{(m-1)} = 0$$

or

$$[(1-x^2)^m P_n^{(m)}]' + (n+m)(n-m+1)(1-x^2)^{m-1}P_n^{(m-1)} = 0. \tag{17.7}$$

From here, in view of the definition (17.1), we come to the recurrence relation (17.5).

Theorem 17.2. *Associated Legendre functions form on the interval* $]-1,1[$ *a set of functions orthogonal with weight function* $\rho(x) = 1$ *with the orthogonality relationship*

$$\int_{-1}^{1} P_k^m(x)P_n^m(x)dx = \frac{2(n+m)!}{(2n+1)(n-m)!}\delta_{nk}. \tag{17.8}$$

Proof. Denote

$$A_{k,n}^m = \int_{-1}^{1} P_k^m(x)P_n^m(x)dx. \tag{17.9}$$

From the definition of the associated Legendre functions we then obtain

$$A_{k,n}^m = \int_{-1}^{1} (1-x^2)^m \Big[\frac{d^m}{dx^m}P_n(x)\Big]\Big[\frac{d^m}{dx^m}P_k(x)\Big]dx.$$

Perform integration by parts, putting

$$U = (1-x^2)^m P_n^{(m)}(x), \quad dV = P_k^{(m)}(x)dx,$$

to get

$$A_{k,n}^m = (1-x^2)^m P_n^{(m)}(x)P_k^{(m-1)}(x)\Big|_{-1}^{1} - \int_{-1}^{1} P_k^{(m-1)}(x)[(1-x^2)^m P_n^{(m)}(x)]'dx. \tag{17.10}$$

Obviously, the term outside the integral sign is equal to zero.

By making use of the recurrence relation (17.7), expression (17.10) can be written as

$$A^m_{k,n} = (n - m + 1)(n + m)\int\limits_{-1}^{1} P_k^{(m-1)}(x)(1 - x^2)^{m-1}P_n^{(m-1)}(x)dx$$

or, according to (17.9),

$$A^m_{k,n} = (n - m + 1)(n + m)A^{m-1}_{k,n}. \tag{17.11}$$

Apply formula (17.11) to $A^{m-1}_{k,n}$, $A^{m-2}_{k,n}$, etc. to get

$$A^m_{k,n} = \frac{(n + m)!}{(n - m)!}A^0_{k,n}. \tag{17.12}$$

In view of (17.9) and (15.2), we have

$$A^0_{k,n} = \|P_n(x)\|^2\delta_{kn} = \frac{2}{2n + 1}\delta_{kn},$$

proving the theorem.

It can be shown that the following is valid:

Statement 17.1. *If a function $f(x)$ satisfies the Dirichlet conditions on the interval $]-1, 1[$, then for this function the expansion*

$$f(x) = \sum_{n=m}^{\infty} C^m_n P^m_n(x), \tag{17.13}$$

is valid, where

$$C^m_n = \frac{2n + 1}{2}\frac{(n - m)!}{(n + m)!}\int\limits_{-1}^{1} f(x)P^m_n(x)dx$$

are Fourier coefficients, and the series (17.13) *itself is called a Fourier series in associated Legendre functions.*

Theorem 17.3. *For the associated Legendre functions with $0 < k, m \leqslant n$, the following orthogonality relationship is valid:*

$$\int\limits_{-1}^{1} P^k_n(x)P^m_n(x)\frac{dx}{1 - x^2} = \frac{1}{m}\frac{(n + m)!}{(n - m)!}\delta_{mk}. \tag{17.14}$$

18 Spherical functions

♦ The functions

$$Y^m_n(\theta, \varphi) = e^{im\varphi}P^m_n(\cos\theta) \tag{18.1}$$

are called *spherical functions* or *spherical harmonics of the first kind* (see Figs. 14–21).

Theorem 18.1. *The functions $Y^m_n(\theta, \varphi)$ satisfy the equation*

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial Y}{\partial\theta}\Big) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2} + n((n + 1)Y = 0. \tag{18.2}$$

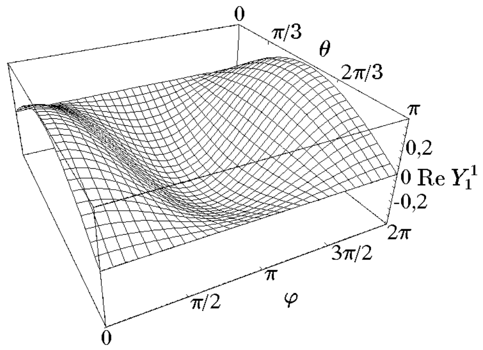

Fig. 14.

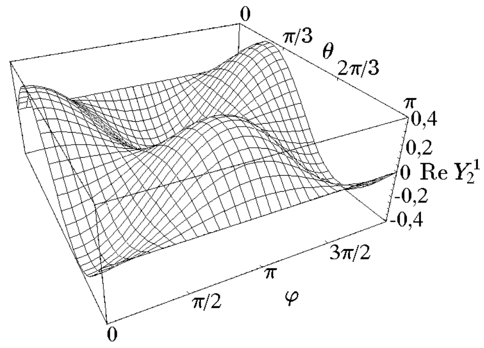

Fig. 15.

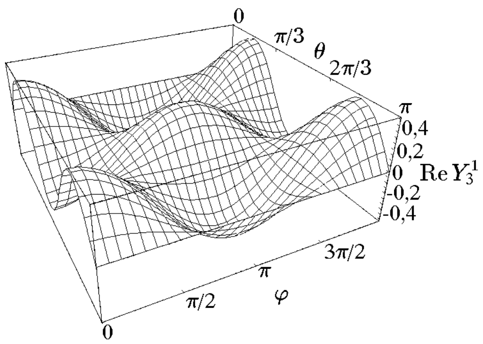

Fig. 16.

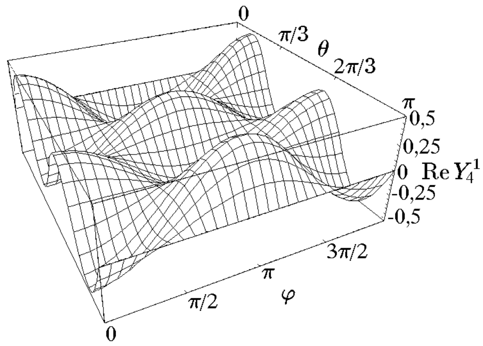

Fig. 17.

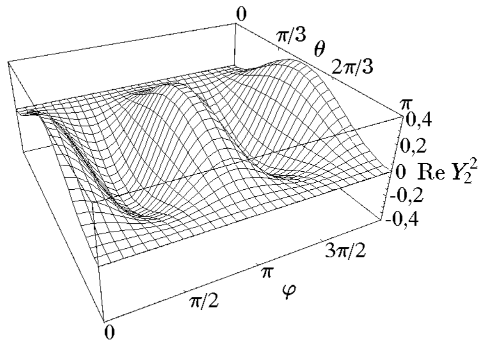

Fig. 18.

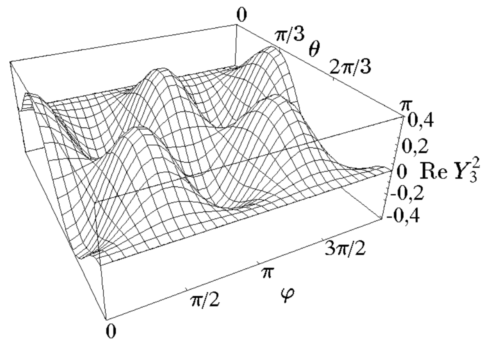

Fig. 19.

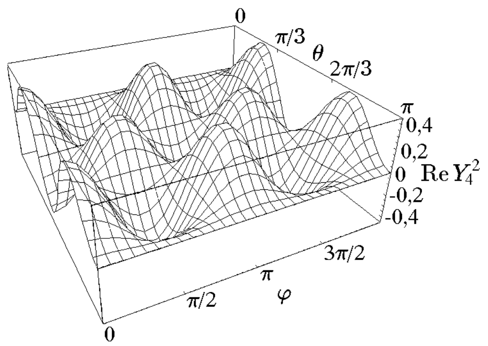

Fig. 20.

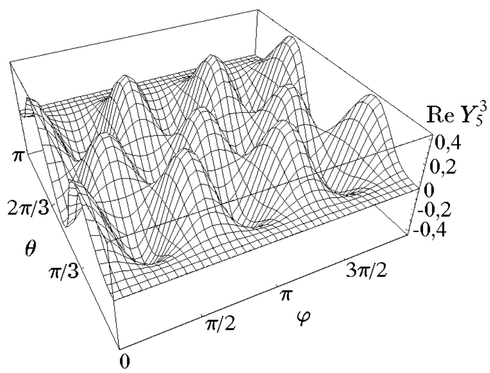

Fig. 21.

Proof. Substitute $Y(\theta,\varphi) = e^{im\varphi}y(\theta)$ into (18.2). Reducing the resulting relation by $e^{im\varphi}$, we obtain for the function $y(\theta)$ the equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\Big(\sin\theta\frac{dy}{d\theta}\Big) + \Big[-\frac{m^2}{\sin^2\theta} + n(n+1)\Big]y = 0. \tag{18.3}$$

Perform in this equation the change of variables $\cos\theta = x$. Then,

$$\sin\theta = \sqrt{1-x^2},$$
$$\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = -\sin\theta\frac{d}{dx} = \sqrt{1-x^2}\frac{d}{dx},$$

and finally we obtain

$$\frac{d}{dx}\Big[(1-x^2)\frac{dy}{dx}\Big] + \Big[n(n+1) - \frac{m^2}{1-x^2}\Big]y = 0. \tag{18.4}$$

Equation (18.4) is Legendre's equation of order m. Hence, the functions $y(x) = P_n^m(x)$ satisfy Eq. (18.4) and the functions (18.1) satisfy Eq. (18.2). Thus, the theorem is proved.

It can be shown (see, e.g., [18], p. 684) that the following theorem is valid:

Theorem 18.2. *The spherical functions $Y_n^m(\theta,\varphi)$ with any φ are eigenfunctions for the Sturm–Liouville problem*

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial Y}{\partial\theta}\Big) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2} + \lambda Y = 0; \tag{18.5}$$
$$|Y(\theta,\varphi)| < \infty, \qquad Y(\theta,\varphi+2\pi) = Y(\theta,\varphi).$$

♦ Bounded solutions of Eq. (18.5) having continuous partial derivatives up to the second order inclusive are called spherical functions.

Theorem 18.3. *The functions $Y_n^m(\theta,\varphi)$ possess the property of orthogonality:*

$$\int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi} Y_n^m(\theta,\varphi)Y_k^l(\theta,\varphi)\sin\theta\,d\theta = \frac{4\pi}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{ml}\delta_{nk}. \tag{18.6}$$

Proof immediately follows from the property of orthogonality of the associated Legendre functions $P_n^m(x)$ (18.4) if one makes the change of variables $x = \cos\theta$ and takes into account the condition

$$\int\limits_0^{2\pi} e^{i(m-l)\varphi}d\varphi = 2\pi\delta_{ml}.$$

19 Hermite polynomials

♦ The function

$$H(x,t) = e^{2xt-t^2} \tag{19.1}$$

is called the generating function for Hermite polynomials.

Theorem 19.1. *The coefficients of the expansion of the function* (19.1) *in a Taylor series in* t

$$H(x,t) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \tag{19.2}$$

are polynomials of degree n.

The polynomials $H_n(x)$ *are called the Hermite polynomials or the Chebyshev–Hermite polynomials.*

Proof. Since the function $H(x,t)$ is analytic in x and t, then

$$H_n(x) = \frac{\partial^n H(x,t)}{\partial t^n}\Big|_{t=0} = \frac{n!}{2\pi i}\oint\limits_{\gamma} \frac{e^{2\omega x-\omega^2}}{\omega^{n+1}}d\omega, \tag{19.3}$$

where γ is an arbitrary closed path enclosing the point $\omega = 0$. But $2\omega x - \omega^2 = x^2 - (x-\omega)^2$, and therefore

$$H_n(x) = \frac{e^{x^2}n!}{2\pi i}\oint\limits_{\gamma} \frac{e^{-(x-\omega)^2}}{\omega^{n+1}}d\omega.$$

Make the change of variables $x - \omega = z$, $\omega = -(z-x)$, $d\omega = -dz$. Then the arbitrary contour γ enclosing the origin will become an arbitrary contour $\tilde{\gamma}$ enclosing the point $z = x$. Then

$$H_n(x) = e^{x^2}\frac{n!}{2\pi i}(-1)^n\oint\limits_{\tilde{\gamma}} \frac{e^{-z^2}}{(z-x)^{n+1}}dz \tag{19.4}$$

or

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\Big[\frac{1}{2\pi i}\oint\limits_{\tilde{\gamma}} \frac{e^{-z^2}}{z-x}dz\Big].$$

According to the Cauchy integral formula,

$$\frac{1}{2\pi i}\oint\limits_{\tilde{\gamma}} \frac{e^{-z^2}}{z-x}dz = e^{-x^2}.$$

Thus,

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \tag{19.5}$$

from which, in view of Leibnitz' formula, it follows that $H_n(x)$ are polynomials of degree n.

♦ Relation (19.5) is called *Rodrigues's formula for the Hermite polynomials.*

Example 19.1. Obtain the explicit form of the Hermite polynomials $H_0(x)$, $H_1(x)$, and $H_2(x)$ and show that

$$H_n(-x) = (-1)^n H_n(x). \tag{19.6}$$

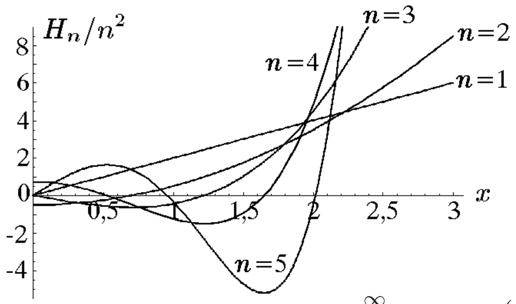

Fig. 22.

Solution. 1. From Rodrigues's formula (19.5) we have

$$H_2(x) = (-1)^2 e^{x^2}(e^{-x^2})'' = e^{x^2}(-2xe^{-x^2})' = \\ = e^{x^2}(-2+4x^2)e^{-x^2} = 4x^2 - 2,$$

that is, $H_2(x) = 4x^2 - 2$. Similarly, we find $H_0(x) = 1$ and $H_1(x) = 2x$ (see Fig. 22).

2. Since $H(-x,-t) = H(x,t)$, we have

$$\sum_{n=0}^{\infty} H(-x)\frac{(-1)^n t^n}{n!} = \sum_{n=0}^{\infty} H(x)\frac{t^n}{n!},$$

which is true if only relation (19.6) holds.

Example 19.2. Calculate $H_n(0)$ for any n.

Solution. We have

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n.$$

Putting $x = 0$, we get

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(0) t^n.$$

On the other hand,

$$e^{-t^2} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{k!}.$$

Then

$$H_{2k+1}(0) = 0, \qquad \frac{1}{(2k)!} H_{2k}(0) = (-1)^k \frac{1}{k!}, \quad k = \overline{0,\infty}$$

or

$$H_{2k}(0) = (-1)^k \frac{(2k)!}{k!}, \qquad k = \overline{0,\infty}.$$

Example 19.3. Show that for the Hermite polynomials the following representation is valid:

$$H_n(x) = \sum_{l=0}^{[n/2]} (-1)^l \frac{n!}{(n-2l)!\,l!} (2x)^{n-2l}. \tag{19.7}$$

Solution. By definition

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n.$$

On the other hand,

$$e^{2xt-t^2} = \sum_{k=0}^{\infty} \frac{1}{k!} (2xt - t^2)^k = \sum_{k=0}^{\infty} \frac{1}{k!} t^k (2x - t)^k =$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} t^k \sum_{l=0}^{k} (-1)^l (2x)^{k-l} t^l C_k^l = \sum_{k=0}^{\infty}\sum_{l=0}^{k} (-1)^l \frac{1}{k!} \frac{k!}{l!(k-l)!} t^{k+l} (2x)^{k-l}.$$

Introduce a new summation index $k+l=n$; then, $k-l=n-2l$. Since $l=\overline{0,k}$, then $n-2l\geqslant 0$, whence $l\leqslant[n/2]$. Thus,

$$e^{2xt-t^2}=\sum_{n=0}^{\infty}\sum_{l=0}^{[n/2]}(-1)^l\frac{1}{l!(n-2l)!}(2x)^{n-2l}t^n.$$

Then,

$$\frac{1}{n!}H_n(x)=\sum_{l=0}^{[n/2]}(-1)^l\frac{1}{l!(n-2l)!}(2x)^{n-2l},$$

$$H_n(x)=\sum_{l=0}^{[n/2]}(-1)^l\frac{n!}{l!(n-2l)!}(2x)^{n-2l}.$$

Example 19.4. Calculate the integral

$$J=\int\limits_{-\infty}^{\infty}\exp(2xy-\alpha^2x^2)H_n(x)dx,\quad \operatorname{Re}\alpha^2>0.$$

Solution. By making use of formula (19.3), we obtain

$$J=\frac{\partial^n}{\partial t^n}\int\limits_{-\infty}^{\infty}\exp(2xy+2xt-\alpha^2x^2-t^2)dx\Big|_{t=0}.$$

Taking into account that

$$\int\limits_{-\infty}^{\infty}\exp(2\beta x-\alpha^2x^2)dx=\frac{\sqrt{\pi}}{\alpha}\exp\frac{\beta^2}{\alpha^2},\quad \operatorname{Re}\alpha^2>0, \tag{19.8}$$

we reduce J to the form

$$J=\frac{\partial^n}{\partial t^n}\frac{\sqrt{\pi}}{\alpha}\exp\Big[\frac{y^2+2yt-(\alpha^2-1)t^2}{\alpha^2}\Big]\Big|_{t=0}.$$

Introducing a variable q as

$$\alpha q=t\sqrt{\alpha^2-1}$$

(the condition $t=0$ leads to $q=0$) and we obtain

$$J=\frac{\sqrt{\pi}(\alpha^2-1)^{n/2}e^{y^2/\alpha^2}}{\alpha^{n+1}}\frac{\partial^n}{\partial q^n}\exp\Big(\frac{2yq}{\alpha\sqrt{\alpha^2-1}}-q^2\Big)\Big|_{q=0}.$$

Applying again relation (19.3), provided that $\operatorname{Re}\alpha^2>0$, we finally find

$$\int\limits_{-\infty}^{\infty}e^{2xy-\alpha^2x^2}H_n(x)dx=\frac{\sqrt{\pi}(\alpha^2-1)^{n/2}}{\alpha^{n+1}}e^{y^2/\alpha^2}H_n\Big(\frac{y}{\alpha\sqrt{\alpha^2-1}}\Big). \tag{19.9}$$

Example 19.5. Show that a Hermite polynomial $H_n(x)$ can be represented in the form

$$H_n(x)=\hat b^n\cdot 1,\qquad \hat b=2x-\frac{d}{dx}. \tag{19.10}$$

Solution. From Rodrigues's formula (19.5), by successive transformations, we obtain

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} =$$
$$= \underbrace{\Big(-e^{x^2}\frac{d}{dx}e^{-x^2}\Big)\Big(-e^{x^2}\frac{d}{dx}e^{-x^2}\Big)\cdots\Big(-e^{x^2}\frac{d}{dx}e^{-x^2}\Big)}_{n \text{ times}} \cdot 1 = \hat{b}^n \cdot 1,$$
$$\hat{b} = -e^{x^2}\frac{d}{dx}e^{-x^2} = 2x - \frac{d}{dx},$$

which is equivalent to formula (19.10).

20 Recurrence relations for Hermite polynomials

Theorem 20.1. *The relations*

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1} = 0, \tag{20.1}$$
$$H_n'(x) = 2nH_{n-1}(x) \tag{20.2}$$

are valid.

Proof. Differentiate Rodrigues's formula (19.5)

$$H_n(x) = (-1)^n e^{x^2}(e^{-x^2})^{(n)}$$

with respect to x:

$$\frac{d}{dx}H_n(x) = (-1)^n 2xe^{x^2}(e^{-x^2})^{(n)} - (-1)^{n+1}e^{x^2}(e^{-x^2})^{(n+1)}.$$

Thus,

$$H_n'(x) = 2xH_n(x) - H_{n+1}(x). \tag{20.3}$$

Now consider

$$H_{n+1}(x) = (-1)^{n+1}e^{x^2}\frac{d^n}{dx^n}\frac{d}{dx}\big(e^{-x^2}\big) = (-1)^n 2e^{x^2}\Big(\frac{d^n}{dx^n}xe^{-x^2}\Big).$$

Since

$$\Big(\frac{d}{dx}\Big)^n xe^{-x^2} = 2(-1)^n\big[x(e^{-x^2})^{(n)} + n(e^{-x^2})^{(n-1)}\big],$$

we obtain

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \tag{20.4}$$

which is equivalent to (20.1). Subtracting (20.4) from (20.3) we obtain (20.2). Thus, the theorem is proved.

Example 20.1. Calculate $H_n'(0)$ for all n.

Solution. Use the recurrence relation

$$H_n'(x) = 2nH_{n-1}(x).$$

Put $x = 0$ to get $H_n'(0) = 2nH_{n-1}(0)$. From here

$$H_{2k}'(0) = 0, \qquad H_{2k+1}'(0) = 2(2k+1)H_{2k}(0).$$

In view of the results of the preceding example, we finally obtain

$$H'_{2k+1}(0) = (-1)^k 2\frac{(2k+1)!}{k!}, \qquad k = \overline{0,\infty}.$$

Theorem 20.2. *Hermite polynomials $y = H_n(x)$ satisfy the differential equation*

$$y''_n(x) - 2xy'_n(x) + 2ny_n(x) = 0 \tag{20.5}$$

or, in selfadjoint form,

$$\frac{d}{dx}\Big(e^{-x^2}\frac{dy_n(x)}{dx}\Big) + 2ne^{-x^2}y_n(x) = 0.$$

Proof. Differentiate (20.2):

$$\frac{d^2H_n(x)}{dx^2} = 2nH'_{n-1}(x).$$

Expressing $H'_{n-1}(x)$ from (20.3), we obtain

$$\frac{d^2H_n(x)}{dx^2} = 2n[2xH_{n-1}(x) - H_n(x)]$$

and writing from (20.2)

$$H_{n-1}(x) = \frac{H'_n}{2n}(x),$$

we find

$$\frac{d^2H_n(x)}{dx^2} = 2xH'_n(x) - 2nH_n(x).$$

The proof is complete.

♦ Equation

$$y'' - 2xy' + \lambda y = 0, \qquad x \in]-\infty,\infty[, \tag{20.6}$$

is called *Hermite's equation.*

Theorem 20.3. *Hermite polynomials $H_n(x)$ are solutions of the Sturm–Liouville problem for Eq. (20.6), provided that the functions $y(x,\lambda)$ are continuous and square-integrable on $]-\infty,\infty[$ with weight function $\rho(x) = e^{-x^2}$ and eigenvalues $\lambda = 2n$.*

21 Orthogonal Hermite polynomials

Theorem 21.1. *For Hermite polynomials, the following orthogonality relationship is valid:*

$$\int\limits_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x)dx = 2^n n!\sqrt{\pi}\delta_{nm}. \tag{21.1}$$

Proof. Let $n \geqslant m$. Consider the integral

$$J = \int\limits_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x)dx = (-1)^n \int\limits_{-\infty}^{\infty} H_m(x)(e^{-x^2})^{(n)}dx.$$

Integrate by parts, putting

$$U = H_m(x),\ dU = H'_m(x)dx,\ dV = (e^{-x^2})^{(n)}dx,\ \text{and } V = (e^{-x^2})^{(n-1)}.$$

Then

$$J = (-1)^n H_m(x)(e^{-x^2})^{(n-1)}\Big|_{-\infty}^{\infty} + (-1)^{n+1}\int_{-\infty}^{\infty} H'_m(x)(e^{-x^2})^{(n-1)}dx.$$

Since $\lim\limits_{|x|\to\infty} H_m(x)e^{-x^2} = 0$, then

$$J = (-1)^{n+1}\int_{-\infty}^{\infty} H'_m(x)(e^{-x^2})^{(n-1)}dx.$$

Perform similar integration $m-1$ times to get

$$J = (-1)^{n+m}\int_{-\infty}^{\infty} H_m^{(m)}(x)(e^{-x^2})^{(n-m)}dx.$$

In virtue of the recurrence relation $H'_m(x) = 2mH_{m-1}(x)$ and the condition $H_0(x) = 1$, we have $H_m^{(m)}(x) = 2^m m!$. Then

$$J = (-1)^{n+m}m!2^m\int_{-\infty}^{\infty}(e^{-x^2})^{(n-m)}dx =$$

$$= (-1)^{n+m}m!2^m\begin{cases} (e^{-x^2})^{(n-m+1)}\Big|_{-\infty}^{\infty} = 0, & n > m\\ \displaystyle\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}, & n = m\end{cases} = n!2^n\sqrt{\pi}\delta_{nm},$$

and finally we obtain

$$J = n!2^n\sqrt{\pi}\delta_{nm},$$

Q. E. D.

It can be shown (see, e.g., [5]) that the following theorem is valid:

Theorem 21.2. *If a function $f(x)$ satisfies the Dirichlet conditions on the interval $]-\infty,\infty[$, then the function $f(x)$ can be expanded in a series in Hermite polynomials*

$$f(x) = \sum_{n=0}^{\infty} C_n H_n(x) \tag{21.2}$$

with coefficients

$$C_n = \frac{1}{\sqrt{\pi}n!2^n}\int_{-\infty}^{\infty} f(x)e^{-x^2}H_n(x)dx, \tag{21.3}$$

and the series (21.2) *converges to the function*

$$f^*(x) = \frac{1}{2}[f(x+0) + f(x-0)].$$

The series (21.2) *is called the Fourier–Hermite series.*

Example 21.1. Expand the function

$$f(x) = \begin{cases} 1 & \text{for} \quad |x| < a, \\ 0 & \text{for} \quad |x| > a \end{cases}$$

in a series in Hermite polynomials.

Solution. Obviously, the conditions of expandability of the function in a Fourier series in Hermite polynomials are fulfilled.

Since the function $f(x)$ is even, the expansion has the form

$$f(x) = \sum_{n=0}^{\infty} C_{2n} H_{2n}(x),$$

where

$$C_{2n} = \frac{1}{2^{2n}(2n)!\sqrt{\pi}} \int_{-a}^{a} e^{-x^2} H_{2n}(x)dx.$$

Make use of the relation

$$\frac{d}{dx}[e^{-x^2} H_{k-1}(x)] = -e^{-x^2} H_k(x), \qquad k = \overline{1, \infty}.$$

Then,

$$C_{2n} = -2\frac{1}{2^{2n}(2n)!\sqrt{\pi}} \int_{0}^{a} \frac{d}{dx}[e^{-x^2} H_{2n-1}(x)] =$$

$$= -\frac{2e^{-a^2}}{2^{2n}(2n)!\sqrt{\pi}} H_{2n-1}(a), \quad n = \overline{1, \infty}.$$

Find the coefficient C_0 immediately as

$$C_0 = \frac{1}{\sqrt{\pi}} \int_{-a}^{a} e^{-x^2} H_0(x)dx = \frac{2}{\sqrt{\pi}} \int_{0}^{a} e^{-x^2} dx = \operatorname{erf} a,$$

where $\operatorname{erf} x$ is the error function (see Sec. "The error integral" of [2]). So,

$$f(x) = \operatorname{erf} a - \frac{2}{\sqrt{\pi}} e^{-a^2} \sum_{n=1}^{\infty} \frac{H_{2n-1}(a)}{2^{2n}(2n)!} H_{2n}(x).$$

Example 21.2. Expand the function

$$\operatorname{sign} x = \begin{cases} -1 & \text{for} \quad x < 0, \\ 0 & \text{for} \quad x = 0, \\ 1 & \text{for} \quad x > 0. \end{cases}$$

in a series in Hermite polynomials.

Solution. The function $y = \operatorname{sign} x$ is odd, and, hence, $C_{2n} = 0$, $n = \overline{0, \infty}$. The expansion has the form

$$\operatorname{sign} x = \sum_{n=0}^{\infty} C_{2n+1} H_{2n+1}(x).$$

Calculate the coefficients of the series. We have

$$C_{2n+1} = \frac{1}{2^{2n}(2n+1)!\sqrt{\pi}} \int\limits_0^\infty e^{-x^2} H_{2n+1}(x)dx.$$

Make use of the relation

$$e^{-x^2} H_{2n+1}(x) = -\frac{d}{dx}[e^{-x^2} H_{2n}(x)].$$

Then,

$$C_{2n+1} = -\frac{1}{2^{2n}(2n+1)!\sqrt{\pi}} e^{-x^2} H_{2n}(x)\Big|_0^\infty = (-1)^n \frac{1}{(2n+1)2^{2n}n!\sqrt{\pi}}.$$

So,

$$\operatorname{sign} x = \frac{1}{\sqrt{\pi}} \sum_{n=0}^\infty (-1)^n \frac{H_{2n+1}(x)}{(2n+1)2^{2n}n!}, \qquad -\infty < x < \infty.$$

Example 21.3. Expand the function $f(x) = e^x$ in a series in Hermite polynomials.

Solution. The sufficient conditions of expandability of the function $f(x)$ in a series in Hermite polynomials are fulfilled. Actually, the function $f(x)$ and its derivative are continuous on any finite interval, and the integral $\int\limits_{-\infty}^{\infty} e^{-x^2} f^2(x)dx$ has a finite value:

$$\int\limits_{-\infty}^{\infty} e^{-x^2} f^2(x)dx = e \int\limits_{-\infty}^{\infty} e^{-(x-1)^2} dx = e\sqrt{\pi}.$$

Then,

$$f(x) = \sum_{n=0}^\infty C_n H_n(x), \qquad -\infty < x < \infty,$$

where

$$C_n = \frac{1}{\|H_n\|^2} \int\limits_{-\infty}^{\infty} e^{-x^2} e^x H_n(x)dx, \quad n = \overline{0, \infty}.$$

Integration by parts, in view of the relation

$$\frac{d}{dx}[e^{-x^2} H_{n-1}(x)] = -e^{-x^2} H_n(x),$$

yields

$$\int\limits_{-\infty}^{\infty} e^{-x^2} e^x H_n(x)dx = -e^x e^{-x^2} H_{n-1}(x)\Big|_{-\infty}^{\infty} + \int\limits_{-\infty}^{\infty} e^x e^{-x^2} H_{n-1}(x)dx.$$

The term outside the integral sign is equal to zero. Actually,

$$\lim_{x\to\infty} e^{x-x^2} H_{n-1}(x) = e^{1/4} \lim_{x\to\infty} \frac{H_{n-1}(x)}{e^{(x-1/2)^2}} = 0.$$

Using integration by parts $n-1$ times, we arrive at the equality

$$C_n = \frac{1}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} e^x H_0(x)dx = \frac{e^{1/4}}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(x-1/2)^2} dx = \frac{e^{1/4}}{2^n n!}.$$

So,

$$e^x = e^{1/4} \sum_{n=0}^{\infty} \frac{H_n(x)}{2^n n!}, \qquad -\infty < x < \infty.$$

◊ The function $f(x) = e^x$ can be expanded in a series in Hermite polynomials by some trick, using the generating function.

Example 21.4. Expand the functions $f(x) = e^{ax}$, $\varphi(x) = \operatorname{sh} ax$, and $g(x) = \operatorname{ch} ax$ in series in Hermite polynomials.

Solution. Put $z = a/2$ in the equality

$$e^{2xz-z^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n$$

to get

$$e^{ax} = e^{a^2/4} \sum_{n=0}^{\infty} \frac{a^n}{2^n n!} H_n(x).$$

In particular,

$$e^x = e^{1/4} \sum_{n=0}^{\infty} \frac{H_n(x)}{2^n n!}.$$

From the last relation, one can easily obtain expansions of the hyperbolic functions:

$$\operatorname{ch} ax = e^{a^2/4} \sum_{n=0}^{\infty} \frac{a^{2n}}{2^{2n}(2n)!} H_{2n}(x),$$
$$\operatorname{sh} ax = e^{a^2/4} \sum_{n=0}^{\infty} \frac{a^{2n+1}}{2^{2n+1}(2n+1)!} H_{2n+1}(x).$$

Example 21.5. Show that

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} H_n^2(x)dx = \sqrt{\pi} 2^n n!(n+1/2).$$

Solution. Use the recurrence relation

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0.$$

Then,

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} H_n^2(x)dx = \frac{1}{4}\|H_{n+1}\|^2 + n^2\|H_{n-1}\|^2 = 2^n n!\sqrt{\pi}\frac{2n+1}{2}.$$

22 Hermite functions

Hermite functions are often used in applications.

♦ A function of the form

$$u_n(x) = \frac{1}{\|H_n(x)\|}\sqrt{\rho(x)}H_n(x), \qquad n = \overline{0, \infty} \tag{22.1}$$

is called *a Hermite function* (see Fig. 23). Here, $\|H_n\|^2 = 2^n n!\sqrt{\pi}$, $\rho(x) = e^{-x^2}$.

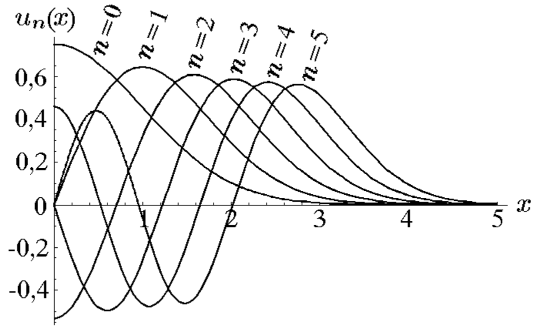

Fig. 23.

Statement 22.1. *Hermite functions $u_n(x)$ are orthonormal on the interval $]-\infty, \infty[$ with weight function $\rho(x) = 1$, that is,*

$$\int\limits_{-\infty}^{\infty} u_n(x)U_m(x)dx = \delta_{nm}. \tag{22.2}$$

Consider

$$\int\limits_{-\infty}^{\infty} u_n(x)u_m(x)dx = \frac{1}{\sqrt{2^{n+m}n!m!\pi}}\int\limits_{\infty}^{\infty} e^{-x^2/2}H_m(x)e^{-x^2/2}H_n(x)dx =$$

$$= \frac{2^n n!\sqrt{\pi}\delta_{nm}}{\sqrt{2^{n+m}n!m!}} = \delta_{nm},$$

which proves the statement.

Theorem 22.1. *Hermite functions $y = u_n(x)$ are the eigenfunctions of the Sturm–Liouville problem*

$$y'' + (\lambda - x^2)y = 0, \qquad \lim_{|x|\to\infty} y(x) = 0, \tag{22.3}$$

corresponding to the eigenvalues $\lambda = 2n+1$ and the normalization condition $\|y(x)\| = 1$.

Proof. Make in the equation the change of variables

$$y(x) = e^{-x^2/2}z(x).$$

Then we obtain for the function $z(x)$ the Sturm–Liouville problem in Hermite polynomials (see Theorem 20.3). In view of formulas (22.1) and (22.2), the theorem is proved.

Example 22.1. Find a tree-term recurrence relation for Hermite functions and express the derivative of a Hermite function in terms of these functions.

Solution. From formula (22.1) we have

$$H_n(x) = (2^n n!\sqrt{\pi})^{1/2} e^{x^2/2} U_n(x). \tag{22.4}$$

Substituting this expression into (20.1), we obtain the three-term recurrence relation

$$\sqrt{\frac{n+1}{2}} u_{n+1}(x) - x u_n(x) + \sqrt{\frac{n}{2}} u_{n-1}(x) = 0. \tag{22.5}$$

Correspondingly, from (20.2), in view of (22.5), we obtain

$$\begin{aligned} u_n'(x) &= \sqrt{2n} u_{n-1}(x) - x u_n(x) = x u_n(x) - \sqrt{2(n+1)} u_{n+1}(x) = \\ &= \sqrt{\frac{n}{2}} u_{n-1}(x) - \sqrt{\frac{n+1}{2}} u_{n+1}(x). \end{aligned} \tag{22.6}$$

The first two relations from (22.6) can be written as

$$\hat{a} u_n(x) = \sqrt{n} u_{n-1}(x), \qquad \hat{a}^+ u_n(x) = \sqrt{n+1} u_{n+1}(x), \tag{22.7}$$

where the operators $\hat{a}$ and $\hat{a}^+$ have the form

$$\hat{a} = \frac{1}{\sqrt{2}}\Big(x + \frac{d}{dx}\Big), \qquad \hat{a}^+ = \frac{1}{\sqrt{2}}\Big(x - \frac{d}{dx}\Big) \tag{22.8}$$

and are called operators of "annihilation" and "birth", respectively. From (22.7) it follows that

$$\begin{gathered} u_n(x) = \frac{1}{\sqrt{n!}} (\hat{a}^+)^n u_0(x), \quad \hat{a}^+ \hat{a} u_n(x) = n u_n(x), \\ \hat{a}\hat{a}^+ u_n(x) = (n+1) u_n(x), \end{gathered} \tag{22.9}$$

that is, all functions $u_n(x)$ "are generated" by the function $u_0(x)$ under the action of the "birth" operator $\hat{a}^+$ of the nth degree and are the eigenfunctions of the operators $\hat{a}^+\hat{a}$ and $\hat{a}\hat{a}^+$. The operators $\hat{a}$ and $\hat{a}^+$ satisfy the permutation relations

$$\hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1, \tag{22.10}$$

which can be verified immediately in view of (22.8).

23 The linear harmonic oscillator

The harmonic oscillator problem is of principal importance in theoretical physics. In studying a complicated system, it is often possible to represent such a system, at least in some approximation, as a set of harmonic oscillators. The harmonic oscillator problem has played a key role, in particular, in developing the second quantization method in quantum field theory, *etc.*

Consider a quantum system with the Hamiltonian

$$\widehat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + \frac{kx^2}{2}, \qquad \hat{p} = -i\hbar\frac{\partial}{\partial x} \tag{23.1}$$

and the problem of finding the admissible energies of the system E and the corresponding wavefunctions $\Psi_E(x)$ that are determined by the stationary Schödinger equation

$$\widehat{\mathcal{H}}\Psi_E = E\Psi_E \tag{23.2}$$

(see also Sec. "Equations of quantum mechanics" of [3]).

The squared modulus of a wavefunction $|\Psi_E(x)|^2$ can be interpreted from the physical viewpoint as the density of the probability that a particle can be detected at the point x. Therefore, the wavefunction $\Psi_E(x)$ should tend to zero as $|x|$ unboundedly increase.

Substituting (23.1) into (23.2), we arrive at the Sturm–Liouville problem

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2}+\frac{kx^2}{2}\Psi=E\Psi, \qquad (23.3)$$
$$\lim_{|x|\to\infty}|\Psi_E(x)|=0.$$

Making the change of variables $x=x_0t$, where

$$x_0=\sqrt{\frac{\hbar}{m\omega_0}},\qquad \omega_0^2=\frac{k}{m},$$

we obtain

$$\frac{d^2\Psi}{dt^2}+\Big(\frac{2mx_0^2E}{\hbar^2}-\frac{kmx_0^4}{\hbar^2}t^2\Big)\Psi=0.$$

Denoting $\lambda=2E/(\hbar\omega_0)$, we obtain the Sturm–Liouville problem for the Hermite functions (22.3). Hence,

$$\lambda_n=2n+1,\qquad \Psi_n(t)=u_n(t).$$

Returning to the original variable, we get

$$E_n=\hbar\omega_0\Big(n+\frac{1}{2}\Big), \qquad (23.4)$$
$$\Psi_n(x)=\frac{1}{\sqrt{x_0}}u_n\Big(\frac{x}{x_0}\Big)=\frac{1}{\sqrt{2^nn!\sqrt{\pi}x_0}}e^{-x^2/(2x_0^2)}H_n\Big(\frac{x}{x_0}\Big).$$

The presence of the multiplier $1/\sqrt{x_0}$ follows from the orthonormality condition.

Example 23.1. The probability for a harmonic oscillator to change from one state to another (which are characterized, respectively, by quantum numbers m and n) is determined by the integral

$$\int_{-\infty}^{\infty} xe^{-x^2}H_n(x)H_m(x)dx.$$

Prove that this integral is equal to

$$\sqrt{\pi}2^{n-1}n!\delta_{m,n-1}+\sqrt{\pi}2^n(n+1)!\delta_{m,n+1},$$

that is, such a transition is possible only between two neighboring energy levels with $m=n\pm1$.

Solution. From the recurrence relation

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$$

we express $xH_n(x)$ and introduce it under the integral. Then we obtain

$$\int\limits_{-\infty}^{\infty} e^{-x^2}\Big(\frac{1}{2}H_{n+1}(x) + nH_{n-1}(x)\Big)H_m(x)dx =$$

$$= \frac{1}{2}\int\limits_{\infty}^{\infty} H_{n+1}(x)H_m(x)e^{-x^2}dx + n\int\limits_{-\infty}^{\infty} H_{n-1}(x)H_m(x)e^{-x^2}dx =$$

$$= 2^n(n+1)!\sqrt{\pi}\delta_{m,n+1} + n2^{n-1}(n-1)!\sqrt{\pi}\delta_{m,n-1}.$$

24 Laguerre polynomials

Define the Laguerre polynomials with the help of a generating function.

♦ The function

$$L^\alpha(x,t) = \frac{1}{(1-t)^{\alpha+1}}e^{-xt/(1-t)}, \quad \alpha > -1, \quad (24.1)$$

is called *the generating function* of the Laguerre polynomials of degree α.

Theorem 24.1. *The coefficients of the expansion of the function $L^\alpha(x,t)$ in a Taylor series in t,*

$$L^\alpha(x,t) = \sum_{n=0}^{\infty} L_n^\alpha(x)t^n, \qquad (24.2)$$

are polynomials of degree n. The polynomials $L_n^\alpha(x)$ are called the (generalized) Laguerre polynomials of order α (see Figs. 24–26).

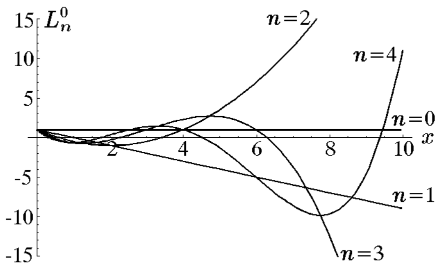

Fig. 24.

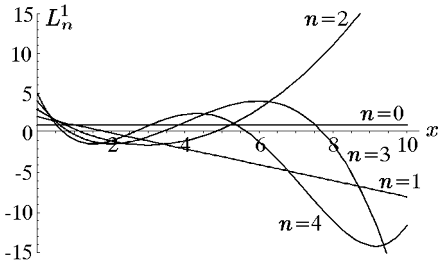

Fig. 25.

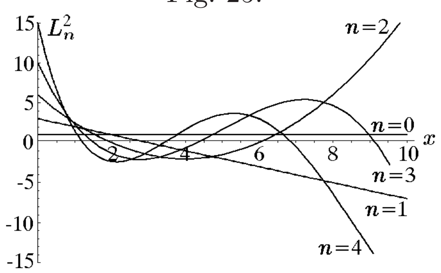

Fig. 26.

Proof. The function $L^\alpha(x,t)$ is analytic in the region $|t| < 1$. Therefore, according to the Cauchy integral formula

$$L_n^\alpha(x) = \frac{1}{n!}\frac{\partial^n L^\alpha(x,t)}{\partial t^n}\Big|_{t=0} = \frac{1}{2\pi i}\oint\limits_{\gamma}\frac{L^\alpha(x,\omega)}{\omega^{n+1}}d\omega,$$

where γ is an arbitrary piecewise smooth path enclosing the point $\omega = 0$. Perform the change of variables

$$\omega = 1 - \frac{x}{z}.$$

Then,

$$d\omega = \frac{x}{z^2}dz, \quad 1-\omega = \frac{x}{z}, \quad \frac{x\omega}{1-\omega} = x\frac{z-x}{z}\frac{z}{x} = z-x,$$

and we obtain

$$L^\alpha(x,\omega) = \frac{z^{\alpha+1}}{x^{\alpha+1}}e^x e^{-z}.$$

Since
$$\omega^{-n-1} = \frac{z^{n+1}}{(z-x)^{n+1}},$$
then
$$L_n^\alpha(x) = \frac{e^x x^{-\alpha}}{2\pi i} \oint\limits_{\tilde{\gamma}} \frac{z^{n+\alpha} e^{-z}}{(z-x)^{n+1}} dz, \tag{24.3}$$
where $\tilde{\gamma}$ is a path enclosing the point $z = x$, or
$$L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} \frac{1}{2\pi i} \oint\limits_{\tilde{\gamma}} \frac{z^{n+\alpha} e^{-z}}{z-x} dz.$$
From Cauchy's integral (I.12.1) we obtain
$$\frac{1}{2\pi i} \oint\limits_{\tilde{\gamma}} \frac{z^{n+\alpha} e^{-z}}{z-x} dz = x^{n+\alpha} e^{-x}.$$
Hence,
$$L_n^\alpha(x) = \frac{1}{n!} \frac{e^x}{x^\alpha} \frac{d^n}{dx^n} [x^{n+\alpha} e^{-x}]. \tag{24.4}$$
Q. E. D.

◊ The Laguerre polynomials of order zero $L_n^0(x)$ are often denoted as $L_n(x)$.

♦ Formula (24.4) is called *Rodrigues's formula for the Laguerre polynomials*.

Example 24.1. Find the explicit form of the Laguerre polynomials $L_0^\alpha(x)$ and $L_1^\alpha(x)$ for any $\alpha > -1$.

Solution. By Rodrigues's formula (24.4),
$$L_0^\alpha(x) = x^{-\alpha} e^x (x^\alpha e^{-x}) = 1,$$
$$L_1^\alpha(x) = x^{-\alpha} e^x (x^{1+\alpha} e^{-x})' = x^{-\alpha} e^x [(1+\alpha) x^\alpha e^{-x} - x^{1+\alpha} e^{-x}] = 1 + \alpha - x.$$
Hence, $L_0^\alpha(x) = 1$, $L_1^\alpha(x) = 1 + \alpha - x$.

Example 24.2. Show that for Laguerre polynomials the following representation is valid:
$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{(-1)^k \Gamma(n+\alpha+1)}{\Gamma(\alpha+k+1) k! (n-k)!} x^k. \tag{24.5}$$

Solution. Perform in relation (24.4) differentiation by Leibniz's formula to get
$$\frac{d^n}{dx^n}(x^{n+\alpha} e^{-x}) = \sum_{k=0}^{n} C_n^k (x^{n+\alpha})^{(n-k)} (e^{-x})^{(k)} =$$
$$= \sum_{k=0}^{n} C_n^k (-1)^k e^{-x} (\alpha+n)(\alpha+n-1)\cdots(\alpha+k+1) x^{\alpha+k}.$$
Then,
$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{(\alpha+n)(\alpha+n-1)\cdots(\alpha+k+1)}{k!(n-k)!} (-x)^k,$$

and, multiplying the numerator and the denominator by $\Gamma(\alpha+k+1)$, in view of the fundamental functional relation for the gamma-function, we obtain

$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)(-x)^k}{k!(n-k)!\Gamma(\alpha+k+1)}.$$

Q. E. D.

25 Recurrence relations for Laguerre polynomials

Theorem 25.1. *The recurrence relation*

$$nL_n^{\alpha-1}(x) = (n+\alpha-1)L_{n-1}^{\alpha-1}(x) - xL_{n-1}^\alpha(x) \tag{25.1}$$

is valid.

Proof. By the definition of the Laguerre polynomials,

$$\begin{aligned} L_n^{\alpha-1}(x) &= \frac{1}{n!}\frac{e^x}{x^{\alpha-1}}\frac{d^n}{dx^n}[x^{n+\alpha-1}e^{-x}] = \\ &= \frac{1}{n!}e^x x^{1-\alpha}\frac{d^{n-1}}{dx^{n-1}}[-e^{-x}x^{n+\alpha-1} + (n+\alpha-1)x^{n+\alpha-2}e^{-x}] = \\ &= -\frac{1}{n!}e^x x^{1-\alpha}[e^{-x}x^{n+\alpha-1}]^{(n-1)} + \frac{n+\alpha-1}{n!}[e^{-x}x^{n+\alpha-2}]^{(n-1)} = \\ &= -\frac{x}{n}L_{n-1}^\alpha(x) + \frac{n+\alpha-1}{n}L_{n-1}^{\alpha-1}(x), \end{aligned}$$

Q. E. D.

Theorem 25.2. *The recurrence relation*

$$L_n^{\alpha+1}(x) = L_{n-1}^{\alpha+1}(x) + L_n^\alpha(x) \tag{25.2}$$

is valid.

Proof. Consider

$$\begin{aligned} L_n^{\alpha+1}(x) &= \frac{1}{n!}e^x x^{-\alpha-1}[e^{-x}x^{n+\alpha+1}]^{(n)} = \\ &= \frac{1}{n!}e^x x^{-\alpha-1}\{x[e^{-x}x^{n+\alpha}]^{(n)} + n[e^{-x}x^{n+\alpha}]^{(n-1)}\} = \\ &= \frac{1}{n!}e^x x^{-\alpha-1+1}[e^{-x}x^{n+\alpha}]^{(n)} + \frac{1}{(n-1)!}e^x x^{-\alpha-1}[e^{-x}x^{n-1+\alpha+1}]^{(n-1)} = \\ &= L_n^\alpha(x) + L_{n-1}^{\alpha+1}(x). \end{aligned}$$

This proves the theorem.

Corollary 25.2.1. *The recurrence relation*

$$(n+1)L_{n+1}^\alpha(x) - (2n+1+\alpha-x)L_n^\alpha(x) + (n+\alpha)L_{n-1}^\alpha(x) = 0 \tag{25.3}$$

is valid.

Proof. Change in (25.2) α by $\alpha-1$ to get

$$L_n^{\alpha-1} = L_n^\alpha(x) - L_{n-1}^\alpha(x). \tag{25.4}$$

Change in (25.1) n by $n+1$. Then,

$$(n+1)L_{n+1}^{\alpha-1}(x) = (n+\alpha)L_n^{\alpha-1}(x) - xL_n^\alpha(x)$$

or, in view of (25.4),

$$(n+1)\big[L_{n+1}^\alpha(x) - L_n^\alpha(x)\big] = (n+\alpha)[L_n^\alpha(x) - L_{n-1}^\alpha(x)] - xL_n^\alpha(x)$$

or

$$(n+1)L_{n+1}^\alpha(x) - (2n+1+\alpha-x)L_n^\alpha(x) + (n+\alpha)L_{n-1}^\alpha(x) = 0.$$

Thus, the theorem is proved.

Corollary 25.2.2. *The recurrence relation*

$$\frac{d}{dx}L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x) = \frac{n}{x}L_n^\alpha(x) - \frac{(n+\alpha)}{x}L_{n-1}^\alpha(x) \tag{25.5}$$

is valid.

Proof. Differentiate relation (24.4) with respect to x to get

$$\frac{dL_n^\alpha(x)}{dx} = \frac{d}{dx}\frac{1}{n!}e^x x^{-\alpha}[e^{-x}x^{\alpha+n}]^{(n)} =$$
$$= \frac{1}{n!}e^x x^{-\alpha}[e^{-x}x^{\alpha+n}]^{(n)} - \frac{\alpha}{n!}e^x x^{-\alpha-1}[e^{-x}x^{\alpha+n}]^{(n)} -$$
$$-\frac{1}{n!}e^x x^{-\alpha}[e^{-x}x^{\alpha+n}]^{(n)} + \frac{\alpha+n}{n!}e^x x^{-\alpha}[e^{-x}x^{\alpha+n-1}]^{(n)}.$$

In view of (24.4), we find

$$\frac{dL_n^\alpha(x)}{dx} = -\frac{\alpha}{x}L_n^\alpha(x) + \frac{n+\alpha}{x}L_n^{\alpha-1}(x).$$

Substitution of (25.4) yields

$$\frac{dL_n^\alpha(x)}{dx} = -\frac{\alpha}{x}L_n^\alpha(x) + \frac{n+\alpha}{x}[L_n^\alpha(x) - L_{n-1}^\alpha(x)] = \frac{n}{x}L_n^\alpha - \frac{n+\alpha}{x}L_{n-1}^\alpha(x)$$

which proves the right side of relation (25.5). Putting in (25.1) $\alpha = \alpha+1$, we get

$$nL_n^\alpha(x) = (n+\alpha)L_{n-1}^\alpha(x) - xL_{n-1}^{\alpha+1}(x).$$

Using the relation obtained, eliminate $nL_n^\alpha(x)$ from the right side of relation (25.5). Then,

$$\frac{dL_n^\alpha(x)}{dx} = \frac{1}{x}\Big[(n+\alpha)L_{n-1}^\alpha(x) - xL_{n-1}^{\alpha+1}(x)\Big] - \frac{n+\alpha}{x}L_{n-1}^\alpha(x) = -L_{n-1}^{\alpha+1}(x),$$

Q. E. D.

Theorem 25.3. *Laguerre polynomials $y = L_n^\alpha(x)$ satisfy the differential equation*

$$xy'' + (\alpha + 1 - x)y' + ny = 0. \tag{25.6}$$

Proof. Let $U(x) = x^{n+\alpha}e^{-x}$. Then,

$$U'(x) = (n+\alpha)x^{n+\alpha-1}e^{-x} - x^{n+\alpha}e^{-x}$$

or

$$xU'(x) + (x - n - \alpha)U(x) = 0.$$

Differentiate this relation $n+1$ times with respect to x. Using Leibnitz's formula

$$(UV)^{(n)} = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} U^{(n-k)}(x)V^{(k)}(x),$$

we obtain

$$x[U^{(n)}(x)]'' + (x + 1 - \alpha)[U^{(n)}(x)]' + (n+1)U^{(n)}(x) = 0. \tag{25.7}$$

Make in this equation the change $U^{(n)} = n!x^\alpha e^{-x}y$. In view of the relations

$$[U^{(n)}(x)]' = n!x^\alpha e^{-x}\Big\{\frac{\alpha - x}{x}y + y'\Big\},$$
$$[U^{(n)}(x)]'' = n!x^\alpha e^{-x}\Big\{\Big[\frac{\alpha(\alpha-1)}{x^2} - \frac{2\alpha}{x} + 1\Big]y + 2\Big[\frac{\alpha}{x} - 1\Big]y' + y''\Big\},$$

dividing (25.7) by the common multiplier $n!x^\alpha e^{-x}$, we obtain

$$x\Big\{\Big[\frac{\alpha(\alpha-1)}{x^2} - \frac{2\alpha}{x} + 1\Big]y + 2\Big[\frac{\alpha}{x} - 1\Big]y' + y''\Big\} +$$
$$+(x + 1 - \alpha)\Big\{\frac{\alpha - x}{x}y + y'\Big\} + (n+1)y = 0.$$

Collecting similar terms, we arrive at the statement of the theorem.

26 Orthogonal Laguerre polynomials

Theorem 26.1. *Laguerre polynomials are orthogonal on the interval* $[0, \infty[$ *with weight function* $\rho(x) = x^\alpha e^{-x}$ *for* $\alpha > -1$. *The orthogonality relationship has the form*

$$\int\limits_0^\infty x^\alpha L_n^\alpha(x)L_m^\alpha(x)e^{-x}dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{nm}. \tag{26.1}$$

Proof. Write Rodrigues's formula for the Laguerre polynomials

$$L_n^\alpha(x) = \frac{1}{n!}e^x x^{-\alpha}[e^{-x}x^{\alpha+n}]^{(n)}.$$

Without loss of generality we assume that $n \geqslant m$. Then,

$$I_{n,m} = \int\limits_0^\infty x^\alpha L_n^\alpha(x)L_m^\alpha(x)e^{-x}dx = \frac{1}{n!}\int\limits_0^\infty L_m^\alpha(x)[e^{-x}x^{\alpha+n}]^{(n)}dx.$$

(a) Let $n = 0$. Then, according to the assumption, $m = 0$. Since $L_0^\alpha(x) = 1$, we have

$$I_{0,0} = \int\limits_0^\infty x^\alpha e^{-x} dx = \Gamma(\alpha + 1). \tag{26.2}$$

(b) Let $n > 0$. Integrate $I_{n,m}$ by parts, putting $U = L_m^\alpha(x)$, $dU = [L_m^\alpha(x)]' dx$, $dV = [e^{-x}x^{\alpha+n}]^{(n)}dx$, and $V = [e^{-x}x^{\alpha+n}]^{(n-1)}$ to get

$$I_{n,m} = \frac{1}{n!}\Big\{L_m^\alpha(x)[e^{-x}x^{\alpha+n}]^{(n-1)}\Big|_0^\infty - \int\limits_0^\infty [e^{-x}x^{\alpha+n}]^{(n-1)}[L_m^\alpha(x)]' dx\Big\}.$$

By definition, $\alpha > -1$ and $n \neq 0$. Therefore, $\alpha + n > 0$, which yields $x^{\alpha+n}|_{x=0} = 0$. Hence, the term outside the integral is equal to zero. Further integration $m-1$ times yields

$$I_{n,m} = \frac{(-1)^n}{n!}\int\limits_0^\infty [e^{-x}x^{\alpha+n}]^{(n-m)}[L_m^\alpha(x)]^{(m)} dx.$$

Calculate the derivative

$$[L_m^\alpha(x)]^{(m)} = \frac{1}{m!}[x^{-\alpha}e^x(x^{\alpha+m}e^{-x})^{(m)}]^{(m)} = \frac{1}{m!}[x^m(-1)^m + Q_{m-1}(x)]^{(m)} = (-1)^m,$$

where $Q_{m-1}(x)$ is a polynomial of degree $m-1$, that is,

$$[L_m^\alpha(x)]^{(m)} = (-1)^m.$$

Then,

$$I_{n,m} = \frac{(-1)^{n+m}}{n!}\int\limits_0^\infty [e^{-x}x^{\alpha+n}]^{(n-m)} dx = \frac{(-1)^{n+m}}{n!}\begin{cases} [e^{-x}x^{\alpha+n}]^{(n-m+1)}\Big|_0^\infty, & n \neq m \\ \int\limits_0^\infty e^{-x}x^{n+\alpha} dx, & n = m \end{cases} =$$

$$= \frac{(-1)^{n+m}}{n!}\begin{cases} 0, & n \neq m; \\ \Gamma(n+\alpha+1), & n = m \end{cases} = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{nm}. \tag{26.3}$$

Combining (26.2) and (26.3), we arrive at the statement of the theorem.

Theorem 26.2. *A Laguerre polynomial is a solution of the eigenvalue problem* (*Sturm–Liouville problem*)

$$xy'' + (\alpha + 1 - x)y' + \lambda y = 0, \qquad \alpha > -1, \tag{26.4}$$

provided that the functions $y(x, \lambda)$ *are continuous, limited at the point* $x = 0$, *and square integrable on* $[0, \infty[$ *with the weight function* $\rho(x) = x^\alpha e^{-x}$. *Equation* (26.4) *is called Laguerre's equation.*

27 Fourier–Laguerre series

Laguerre polynomials, as the above-discussed orthogonal polynomials, can be used as a base for the expansion of a function $f(x)$ in a functional series. For this case, the following theorem can be proved:

Theorem 27.1. *If a function $f(x)$ satisfies the Dirichlet conditions on the interval $[0, \infty[$, i.e., $\left|\int\limits_0^\infty f^2(x)dx\right| < \infty$, is piecewise continuous on this interval, and has discontinuities of the first kind only, this function can be expanded in a series in Laguerre polynomials with coefficients*

$$C_n^\alpha = \frac{n!}{\Gamma(n+\alpha+1)} \int\limits_0^\infty x^\alpha e^{-x} L_n^\alpha(x)dx, \qquad \alpha > -1, \tag{27.1}$$

and the series converges:

$$\sum_{n=0}^\infty C_n^\alpha L_n^\alpha(x) = \frac{1}{2}[f(x+0) + f(x-0)]. \tag{27.2}$$

The series (27.2) is called the Fourier–Laguerre series.

Example 27.1. Expand the function $f(x) = e^{-bx}$ $(b > 0)$ in a Fourier series in Laguerre polynomials.

Solution. The conditions of expandability of the function are fulfilled. Hence,

$$e^{-bx} = \sum_{n=0}^\infty C_n^\alpha L_n^\alpha(x), \quad C_n^\alpha = \frac{n!}{\Gamma(n+\alpha+1)} \int\limits_0^\infty e^{-x} x^\alpha e^{-bx} L_n^\alpha(x)dx.$$

Find C_n^α, $n = \overline{0, \infty}$. By Rodrigues's formula for Laguerre polynomials (24.4),

$$L_n^\alpha(x) = \frac{1}{n!} x^{-\alpha} e^x \frac{d^n(x^{n+\alpha}e^{-x})}{dx^n}.$$

Then,

$$C_n^\alpha = \frac{1}{\Gamma(n+\alpha+1)} \int\limits_0^\infty e^{-bx} \frac{d^n(x^{n+\alpha}e^{-x})}{dx^n} dx.$$

Integration by parts yields

$$I_n = \int\limits_0^\infty e^{-bx} \frac{d^n(x^{n+\alpha}e^{-x})}{dx^n} dx = e^{-bx} \frac{d^{n-1}(x^{n+\alpha}e^{-x})}{dx^{n-1}}\Big|_0^\infty + b \int\limits_0^\infty e^{-bx} \frac{d^{n-1}(x^{n+\alpha}e^{-x})}{dx^{n-1}} dx.$$

For $x = \infty$ the term outside the integral is equal to zero due to the multiplier e^{-bx} $(b > 0)$, and for $x = 0$ it is equal to zero due to the second multiplier

$$\frac{d^{n-1}(x^{n+\alpha}e^{-x})}{dx^{n-1}},$$

since, if we calculate the derivative of the $n-1$th order by Leibniz's formula, we obtain a sum with each term containing x to a positive power (with the lowest power equal to $\alpha + 1$). However, by the definition of Laguerre polynomials, $\alpha > -1$ and hence $\alpha + 1 > 0$.

Integration by parts n times yields the equality

$$\int_0^\infty e^{-bx}\frac{d^n(x^{n+\alpha}e^{-x})}{dx^n}dx = b^n\int_0^\infty e^{-(b+1)x}x^{n+\alpha}dx.$$

Perform in the integral the change of variables $(b+1)x = t$ to get

$$I_n = \frac{b^n}{(b+1)^{n+\alpha+1}}\int_0^\infty e^{-t}t^{n+\alpha}dt = \frac{b^n\Gamma(n+\alpha+1)}{(b+1)^{n+\alpha+1}}.$$

Then,

$$C_n^\alpha = \frac{I_n}{\Gamma(n+\alpha+1)} = \frac{b^n}{(b+1)^{n+\alpha+1}}, \qquad n = \overline{0,\infty}.$$

Find the expansion

$$e^{-bx} = \frac{1}{(b+1)^{\alpha+1}}\sum_{n=0}^\infty\Big(\frac{b}{b+1}\Big)^n L_n^\alpha(x), \qquad 0 < x < \infty.$$

Consider a particular case of $b = 1$. Then,

$$e^{-x} = \frac{1}{2^{\alpha+1}}\sum_{n=0}^\infty\frac{1}{2^n}L_n^\alpha(x), \qquad 0 < x < \infty.$$

If $\alpha = 0$, then

$$e^{-x} = \frac{1}{2}\sum_{n=0}^\infty\frac{1}{2^{n+1}}L_n(x), \qquad 0 < x < \infty. \tag{27.3}$$

Example 27.2. Expand the function $f(x) = x^p$, $x > 0$, in a Fourier series in Laguerre polynomials.

Solution. For the Fourier coefficients, we have the representation

$$C_n^\alpha = \frac{n!}{\Gamma(n+\alpha+1)}\int_0^\infty e^{-x}x^\alpha x^p L_n^\alpha(x)dx, \qquad n = \overline{0,\infty}$$

or

$$C_n^\alpha = \frac{1}{\Gamma(n+\alpha+1)}\int_0^\infty x^p\frac{d^n}{dx^n}(x^{n+\alpha}e^{-x})dx, \qquad n = \overline{0,\infty}.$$

As in Example 27.1, integrate the last expression by parts n times to get

$$C_n^\alpha = \frac{(-1)^n p(p-1)\cdots(p-n+1)}{\Gamma(n+\alpha+1)}\int_0^\infty x^{p-n}x^{n+\alpha}e^{-x}dx;$$

$$C_n^\alpha = \frac{(-1)^n p(p-1)\cdots(p-n+1)}{\Gamma(n+\alpha+1)}\Gamma(p+\alpha+1).$$

Then for $0 < x < \infty$

$$x^p = \Gamma(p+\alpha+1)\sum_{n=0}^\infty(-1)^n\frac{p(p-1)\cdots(p-n+1)}{\Gamma(n+\alpha+1)}L_n^\alpha(x). \tag{27.4}$$

Example 27.3. Expand the function $f(x) = \ln x$ in a series in Laguerre polynomials.

Solution. The sought-for expansion has the form

$$f(x) = \sum_{n=0}^{\infty} C_n^\alpha L_n^\alpha(x),$$

where

$$C_n^\alpha = \frac{n!}{\Gamma(n+\alpha+1)} \int_0^\infty x^\alpha e^{-x} L_n^\alpha(x) \ln x \, dx =$$
$$= \frac{1}{\Gamma(n+\alpha+1)} \int_0^\infty (x^{n+\alpha} e^{-x})^{(n)} \ln x \, dx. \tag{27.5}$$

Consider the case $n \geqslant 1$. Integrate (27.5) by parts, putting $U = \ln x$, $dU = dx/x$, $dV = (x^{n+\alpha}e^{-x})^{(n)}dx$, and $V = (x^{n+\alpha}e^{-x})^{(n-1)}$ to get

$$C_n^\alpha = -\frac{1}{\Gamma(n+\alpha+1)} \int_0^\infty \frac{1}{x} (x^{n+\alpha} e^{-x})^{(n-1)} dx.$$

Repeating this $n-1$ times, we obtain

$$C_n^\alpha = -\frac{(n-1)!}{\Gamma(n+\alpha+1)} \int_0^\infty \frac{1}{x^n} x^{n+\alpha} e^{-x} dx = -\frac{(n-1)!\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}.$$

If $n = 0$, then

$$C_0^\alpha = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty x^\alpha e^{-x} \ln x dx = \frac{\Gamma'(\alpha+1)}{\Gamma(\alpha+1)}.$$

Hence,

$$\ln x = \frac{\Gamma'(\alpha+1)}{\Gamma(\alpha+1)} - \Gamma(\alpha+1) \sum_{n=1}^{\infty} \frac{(n-1)!}{\Gamma(n+\alpha+1)} L_n^\alpha(x).$$

PART IV

Equations of Mathematical Physics

This section is the core of the whole course of mathematical physics. The overwhelming majority of physical problems can be formulated mathematically in the form of various differential or integral equations. It turns out that physical problems which, at first glance, are quite dissimilar can be described by formally identical mathematical equations. We have thought it useful to demonstrate this by a great number of various examples illustrating not only the way of constructing equations, but also the character of initial and boundary conditions. It is the most typical equations that are the subject matter of the given part of the course. The properties of solutions of these equations are formulated as theorems the proofs of which, with rare exception, are given. The main properties of the solutions are illustrated by problems. In the majority of cases, the solutions of the problems are followed by plots providing some information about the character of the solutions.

CHAPTER 1

First order partial differential equations

Introduction

We shall consider the partial differential equations that describe mathematical models of physical phenomena. It is these equations that are called differential equations of mathematical physics.

♦ An equation which contains, besides independent variables $\vec{x} \in \mathbb{R}^n$ and the sought-for function $u = u(\vec{x})$, partial derivatives of this function is called a partial differential equation.

Such an equation is written in the form

$$F(x_1, x_2, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1x_1}, \ldots, u_{x_kx_n\ldots}, \ldots) = 0, \qquad (0.1)$$

where F is a given function of its arguments. Here and below we use the notations

$$u_{x_k} = \frac{\partial u}{\partial x_k}, \qquad u_{x_k,x_l} = \frac{\partial^2 u}{\partial x_k \partial x_l}, \ldots$$

♦ The order of such an equation is equal to the order of the higher derivative involved in the equation.

♦ Any function $u = \varphi(x_1, \ldots, x_n)$ which, being substituted into Eq. (0.1), turns it into an identity is called a solution of this equation.

♦ The totality of all particular solutions of Eq. (0.1) is called the general solution of Eq. (0.1).

Example 0.1. Find the general solution of the equation

$$\frac{\partial u}{\partial x} = y + x, \qquad (0.2)$$

where $u = u(x, y)$.

Solution. Integrate (0.2) with respect to x to get

$$u(x,y) = xy + \frac{x^2}{2} + \Phi(y),$$

where $\Phi(y)$ is an arbitrary function.

Example 0.2. Find the general solution of the equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} = 0. \tag{0.3}$$

Solution. Integrate the equation one time with respect to the variable x to get

$$\frac{\partial u(x,y)}{\partial x} = \varphi(y),$$

where $\varphi(y)$ is an arbitrary function of the variable y. Integrating again with respect to x, we find

$$u(x,y) = x\varphi(y) + \psi(y),$$

where $\psi(y)$ is an arbitrary function as well.

♢ From the above examples it can be seen that the general solutions of Eqs. (0.2) and (0.3) involve arbitrary functions. Therein lies the difference between the general solution of a partial differential equation and the general solution of an ordinary differential equation which contains as a rule only arbitrary constants whose number is equal to the order of the equation. For a first order partial differential equation, a general solution depending on an arbitrary function commonly exists, while for a second order partial differential equation, the general solution depends on two arbitrary functions. However, for many partial differential equations of higher orders, it is difficult to represent the general solution in terms of arbitrary functions.

1 Linear first order partial differential equations

♦ A first order partial differential equation is an equation containing an unknown function and its partial derivatives of the first order only..

The general form of a first order partial differential equation is

$$F(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_n}) = 0 \tag{1.1}$$

or, in vector form,

$$F(\vec{x}, u, \nabla u) = 0, \qquad \vec{x} \in \mathbb{R}^n_x. \tag{1.2}$$

Here F is a given function of its own arguments.

♢ Geometrically, the solution $u = u(\vec{x})$ of Eq. (1.2) can be interpreted as a surface in an $n+1$-dimensional space $(\vec{x}, u) \in \mathbb{R}^{n+1}_{x,u}$, which is the direct product of the space $\mathbb{R}^n_x$ by the space $\mathbb{R}_u$ ($\mathbb{R}^n_x \times \mathbb{R}_u$). This surface is called the integral surface of Eq. (1.2).

♦ An equation of the form

$$\sum_{k=1}^{n} \frac{\partial u}{\partial x_k} a_k(\vec{x}) + u a_0(\vec{x}) = b(\vec{x}), \tag{1.3}$$

where $b(\vec{x}) = b(x_1, \ldots, x_n)$ and $a_k(\vec{x}) = a_k(x_1, \ldots, x_n)$, $k = \overline{0, n}$, are given functions, is called a linear first order partial differential equation. Equation (1.3) is called homogeneous if $b(\vec{x}) \equiv 0$; otherwise it is nonhomogeneous.

♦ An equation of the form

$$\sum_{k=1}^{n}\frac{\partial u}{\partial x_k}a_k(\vec{x},u)=a_0(\vec{x},u) \tag{1.4}$$

is called quasi-linear (linear with respect to partial derivatives). If $a_0(\vec{x},u)=0$, Eq. (1.4) is called quasi-linear homogeneous; otherwise it is nonhomogeneous.

Let us consider in more detail quasi-linear equations with two independent variables:

$$P(x,y,u)u_x+Q(x,y,u)u_y=R(x,y,u). \tag{1.5}$$

The functions P, Q, and R are assumed to be continuously differentiable in some region D of the space $\mathbb{R}^3_{x,y,u}$ with P and Q not vanishing simultaneously in the region D.

The given functions (P,Q,R) define in the region D a vector field $\vec{a}=(P,Q,R)$.

♦ A curve L is called the vector line of a field $\vec{a}$ or the integral curve corresponding to this direction field if at each point of this curve the tangent vector is parallel to the vector $\vec{a}$.

The vector lines of the field $\vec{a}$ are determined by integrating the system of ordinary differential equations

$$\frac{dx}{P(x,y,u)}=\frac{dy}{Q(x,y,u)}=\frac{du}{R(x,y,u)}=dt. \tag{1.6}$$

We have already mentioned that a solution of Eq. (1.5) defines in the space x,y,u some surface $u=u(x,y)$ which is called integral. The normal to this surface is parallel to the vector $\vec{n}=(u_x,u_y,-1)$. In this case, Eq. (1.5) expresses the condition that the normal is orthogonal to the vector lines of the field of directions:

$$(\vec{n},\vec{a})=0,\qquad \vec{n}=(u_x,u_y,-1),\qquad \vec{a}=(P,Q,R). \tag{1.7}$$

The system of equations (1.6), written as

$$\begin{cases}\dfrac{dx}{dt}=\dot{x}=P(x,y,u),\\ \dfrac{dy}{dt}=\dot{y}=Q(x,y,u),\\ \dfrac{du}{dt}=\dot{u}=R(x,y,u),\end{cases} \tag{1.8}$$

specifies in parametric form (with t being a parameter) the vector lines of the field $\vec{a}$.

♦ The system of equations (1.6) or (1.8) is called *characteristic* and its solutions are called characteristics of Eq. (1.5).

◊ If the surface $u=u(x,y)$ is the geometric locus of the characteristic lines of Eq. (1.5), that is, it is formed by the lines satisfying the system of equations (1.8), then any plane tangential to this surface is orthogonal to the vector $\vec{n}$. Hence, the function $u(x,y)$, which specifies the surface, satisfies Eq. (1.5) and the surface $u=u(x,y)$ is the integral surface of this equation.

Let us consider the first integral of the characteristic system of equations (1.8).

♦ A function $\Psi(x_1,\ldots,x_n,t)$ which is not identically equal to a constant, but holds a constant value on solutions $\vec{x}=\vec{x}(t)$ of Eqs. (1.9), i.e., $\Psi(x_1(t),\ldots,x_n(t),t)=C$, is called the first integral of the system of differential equations

$$\dot{x}_j=f_j(x_1,\ldots,x_n,t),\qquad j=\overline{1,n}. \tag{1.9}$$

Theorem 1.1. *Let*

$$\Psi(x, y, u) = C \tag{1.10}$$

be the first integral of the system of equations (1.8), $\Psi(x, y, u)$ *be differentiable with respect to all its arguments, and*

$$\frac{\partial \Psi}{\partial u}(x, y, u) \neq 0.$$

Then the function $u = \varphi(x, y)$ *implicitly defined by relation* (1.10) *satisfies Eq.* (1.5).

Proof. The function $\Psi(x, y, u) = C$ is the first integral of the system of equations (1.8). Hence,

$$\frac{d}{dt}[\Psi(x, y, u) - C] = 0,$$

that is,

$$\frac{\partial \Psi}{\partial x}\frac{dx}{dt} + \frac{\partial \Psi}{\partial y}\frac{dy}{dt} + \frac{\partial \Psi}{\partial u}\frac{du}{dt} = 0.$$

Divide the obtained equation by $\partial\Psi/\partial u$. Taking into account the system of equations (1.8) and the rules for the differentiation of implicitly specified functions,

$$\frac{\partial u}{\partial x} = -\frac{\partial \Psi}{\partial x}\Big/\frac{\partial \Psi}{\partial u}, \quad \frac{\partial u}{\partial y} = -\frac{\partial \Psi}{\partial y}\Big/\frac{\partial \Psi}{\partial u},$$

we obtain

$$-\frac{\partial u}{\partial x}P(x, y, z) - \frac{\partial u}{\partial y}Q(x, y, u) + R(x, y, u) = 0.$$

This proves the theorem.

Corollary. Let

$$\begin{array}{l} \Psi_1(u, x, y) = C_1, \\ \Psi_2(u, x, y) = C_2 \end{array} \tag{1.11}$$

be two linearly independent first integrals of the system (1.8). Then the general solution of Eq. (1.5) is implicitly specified by the relation

$$\Phi(\Psi_1(u, x, y), \Psi_2(u, x, y)) = 0, \tag{1.12}$$

where $\Phi(C_1, C_2)$ is an arbitrary smooth function of two variables.

Proof is similar to that of Theorem 1.1.

Example 1.1. Find the general solution of the differential equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1.$$

Solution. Note that $\vec{a} = (1, 1, 1)$. Hence, the characteristic system of equations has the form $dx = dy = du$ or

$$\begin{cases} dx = dy, \\ dx = du, \end{cases} \quad \text{which yields} \quad \begin{cases} x - y = C_1, \\ x - u = C_2. \end{cases}$$

In virtue of the above corollary, the general solution is implicitly specified by the equation

$$\Phi(x - y, x - u) = 0.$$

Since the function u appears only in one first integral, the general solution of the equation can be written in the form

$$\widetilde{\Phi}(x-y)+x-u=0$$

or

$$u(x,y)=\widetilde{\Phi}(x-y)+x,$$

where $\widetilde{\Phi}(\omega)$ is an arbitrary function.

♢ The Cauchy problem for the differential equation (1.5) is formulated as follows: determine the integral surface of Eq. (1.5) which passes through a given curve in the space (u,x,y).

In implicit form, this curve is given by the system of equations

$$\begin{cases}\Phi_1(u,x,y)=0,\\ \Phi_2(u,x,y)=0\end{cases} \tag{1.13}$$

on condition that

$$\operatorname{rang}\begin{pmatrix}\dfrac{\partial\Phi_1}{\partial u} & \dfrac{\partial\Phi_1}{\partial x} & \dfrac{\partial\Phi_1}{\partial y}\\ \dfrac{\partial\Phi_2}{\partial u} & \dfrac{\partial\Phi_2}{\partial x} & \dfrac{\partial\Phi_2}{\partial y}\end{pmatrix}=2.$$

♢ The above theorems can readily be generalized for the case of the quasi-linear equation (1.4) with an arbitrary number of variables. These generalizations are completely described by Theorem 1.2 and Theorem 1.3 prior to which it would be useful to consider some examples.

Example 1.2. Find the general solution of the equation

$$x\frac{\partial u}{\partial x}-yz\frac{\partial u}{\partial z}=0,\quad u=u(x,y,z) \tag{1.14}$$

and select from the general solution a particular one satisfying the condition

$$u\big|_{z=1}=x^y. \tag{1.15}$$

Solution. 1. In our case, Eq. (1.14) is represented in the form

$$P(x,y,z)u_x+Q(x,y,z)u_y+R(x,y,z)u_z=Z(x,y,z),$$

where

$$P(x,y,z)=x,\quad Q(x,y,z)=0,$$
$$R(x,y,z)=-yz,\quad Z(x,y,z)=0.$$

Hence, the system of characteristic equations

$$\frac{dx}{P(x,y,z)}=\frac{dy}{Q(x,y,z)}=\frac{dz}{R(x,y,z)}=\frac{du}{Z(x,y,z)}$$

becomes

$$\frac{dx}{x}=\frac{dy}{0}=\frac{dz}{-yz}=\frac{du}{0}. \tag{1.16}$$

Here, the zero in the denominator is meant in the sense of proportion; that is, if we write $a/0=b/c$, then $a=(b/c)\cdot 0=0$. From (1.16) we obtain

$$dy=0,\qquad du=0,\qquad \frac{dx}{x}=\frac{dz}{-yz}$$

and find the first integrals

$$y = C_1, \qquad u = C_3, \qquad \ln x = -\frac{\ln z}{C_1} + \ln C_2$$

or

$$C_2 = xz^{1/C_1} = xz^{1/y}.$$

Hence, the general solution of Eq. (1.14) is determined implicitly by the equation

$$\Phi(y, xz^{1/y}, u) = 0, \tag{1.17}$$

where $\Phi(y, \omega, u)$ is an arbitrary function of three variables. Solving Eq. (1.17) for u, we obtain

$$u = f(y, \omega)\big|_{\omega = xz^{1/y}},$$

where $f(y, \omega)$ is an arbitrary function of two variables. From condition (1.15) we find

$$u\big|_{z=1} = f(y, \omega)\big|_{\omega = x} = x^y.$$

Expressing the right side of this relation in terms of ω, we obtain that

$$f(y, \omega) = \omega^y.$$

Hence, the solution of the problem (1.14), (1.15) is

$$u = \omega^y\big|_{\omega = xz^{1/y}} = (xz^{1/y})^y = zx^y. \tag{1.18}$$

Let us generalize the results obtained for a multidimensional case. Let us consider a nonlinear system of first order differential equations

$$\dot{\vec{x}} = \vec{a}(\vec{x}, u), \qquad \dot{u} = a_0(\vec{x}, u). \tag{1.19}$$

Assume that the functions $\vec{a}(\vec{x}, u)$ and $a_0(\vec{x}, u)$ are continuously differentiable in a region $D \subset \mathbb{R}^{n+1}_{\vec{x},u}$, and the vector $\vec{a}(\vec{x}, u)$ is other than the identical zero in the region D.

◊ Further we will use the notation

$$\langle \vec{a}, \vec{b} \rangle = \sum_{k=1}^{n} a_k b_k$$

for an Euclidean scalar product in contrast to a Hermitian scalar product

$$(\vec{a}, \vec{b}) = \sum_{k=1}^{n} a_k^* b_k.$$

Theorem 1.2. *Let $w(\vec{x}, u)$ be the first integral of the system of equations* (1.19). *Then the function $u(\vec{x}, t)$, which is determined implicitly by the equation $w(\vec{x}, u) = 0$, satisfies the quasi-linear equation*

$$\langle \vec{a}(\vec{x}, u), \nabla u \rangle = a_0(\vec{x}, u). \tag{1.20}$$

Proof. By the rule of the differentiation of composite functions we find

$$\nabla u = -\frac{\nabla w}{\partial w / \partial u}. \tag{1.21}$$

Differentiate the first integral of the system of equations (1.20) with respect to t:

$$\frac{dw}{dt} = \langle \nabla w, \dot{\vec{x}} \rangle + \frac{\partial w}{\partial u}\dot{u} = \langle \nabla w, \vec{a}(\vec{x}, u) \rangle + \frac{\partial w}{\partial u} a_0(\vec{x}, u) = 0.$$

Making use of relation (1.21), we obtain (1.20), Q. E. D.

Theorem 1.3. *Let $w_k(\vec{x}, u)$ and $k = \overline{1, n}$ be independent first integrals of the system of equations* (1.19). *Then the general solution of Eq.* (1.20) *is determined by the relation*

$$F\big(w_1(\vec{x}, u), w_2(\vec{x}, u), \ldots, w_n(\vec{x}, u)\big) = 0, \tag{1.22}$$

where $F(w_1, w_2, \ldots, w_n)$ is an arbitrary smoothly varying function of n variables.

Proof is similar to that of Theorem 1.1.

♦ The system of equations (1.19) is called the system of characteristic equations for Eq. (1.20).

2 The Cauchy problem for linear first order partial differential equations

Consider the equation

$$\langle \vec{a}(\vec{x}), \nabla u \rangle + a_0(\vec{x})u = f(\vec{x}). \tag{2.1}$$

The Cauchy problem for Eq. (2.1) is formulated for a surface of dimension $n - 1$ in the space $\mathbb{R}^n_{\vec{x}}$.

♦ A set in $\mathbb{R}^n$, which is specified by an equation of the form

$$\vec{x} = \vec{\varphi}(s_1, \ldots, s_{n-1}), \qquad (s_1, \ldots, s_{n-1}) \in U,$$

where U is a region in the space $\mathbb{R}^{n-1}_s$, and $\vec{\varphi}(s_1, \ldots, s_{n-1})$ is a vector function continuously differentiable in U and

$$\operatorname{rang} \left\| \frac{\partial \varphi_i}{\partial s_k} \right\| = n - 1$$

is called a *smooth hypersurface* γ.

♦ The Cauchy problem for Eq. (2.1) is the problem of finding the solution to this equation, which satisfies the condition

$$u(\vec{x})|_\gamma = h(s_1, \ldots, s_{n-1}), \tag{2.2}$$

where $h(s_1, \ldots, s_{n-1})$ is a function continuously differentiable in U. The surface γ is called *the Cauchy surface.*

♦ The system of equations

$$\dot{\vec{x}} = \vec{a}(\vec{x}) \tag{2.3}$$

is called *the characteristic system of equations* for Eq. (2.1) and its solution is called *a characteristic.*

Theorem 2.1. *Assume that a hypersurface γ does not touch the characteristics. Then the Cauchy problem* (2.1), (2.2) *is uniquely solvable within a certain neighborhood of the hypersurface γ.*

Proof. 1. Assume that from every point of the surface γ comes a characteristic of equations (2.3), that is, solve the Cauchy problem for the system of equations (2.3):

$$\vec{x}|_{t=0} = \vec{\varphi}(s_1, \ldots, s_{n-1}), \qquad (s_1, \ldots, s_{n-1}) \in U. \tag{2.4}$$

2. Let

$$\vec{x} = \vec{X}(t, s_1, \ldots, s_{n-1}) \tag{2.5}$$

be a solution to the Cauchy problem (2.3), (2.4). Then along the characteristics

$$\frac{du}{dt} = \langle \nabla u, \dot{\vec{x}} \rangle = \langle \nabla u, \vec{a}(\vec{x}) \rangle. \tag{2.6}$$

3. To determine the function $u = U(t, s_1, \ldots, s_{n-1})$, we have the Cauchy problem for the characteristics of Eqs. (2.3)

$$\frac{du}{dt} + a_0(\vec{x}(t))u = f(\vec{x}(t)), \qquad u|_{t=0} = h(s_1, \ldots, s_{n-1}). \tag{2.7}$$

Solving this equation, we obtain $u = U(t, s_1, \ldots, s_{n-1})$ as a smooth function of $t, s_1, \ldots,$ $\ldots, s_{n-1}$.

4. Solve the system of Eqs. (2.5) with respect to t and s_k, that is, find $t = T(\vec{x})$ and $s_k = S_k(\vec{x})$, $k = \overline{1, n-1}$. As a result, the solution of the Cauchy problem (2.1)–(2.5) will be written in the form

$$u(\vec{x}) = U\big(T(\vec{x}), S_1(\vec{x}), \ldots, S_{n-1}(\vec{x})\big).$$

It remains to show that U is a smooth function of the variable $\vec{x}$. To do this, it suffices to check that one can express $t, s_1, \ldots, s_{n-1}$ from relation (2.5) as smooth functions of $\vec{x}$.

Actually, the Jacobian

$$J = \left|\frac{\partial \vec{X}}{\partial t}, \frac{\partial \vec{X}}{\partial s_1}, \ldots, \frac{\partial \vec{X}}{\partial s_{n-1}}\right|\Big|_{t=0} = \left|\vec{a}(\vec{x}), \frac{\partial \vec{x}}{\partial s_1}, \ldots, \frac{\partial \vec{x}}{\partial s_{n-1}}\right|_{\vec{x}\in\gamma} \neq 0,$$

since, as agreed, the hypersurface γ does not touch the characteristics. Thus, we have proved the existence of a solution of the Cauchy problem.

5. Assume that the Cauchy problem (2.1), (2.2) has two solutions: $u_1(\vec{x})$ and $u_2(\vec{x})$. Introduce $v = u_1 - u_2$ to get

$$\langle \nabla v, \vec{a}(\vec{x}) \rangle + a_0(\vec{x})v = 0, \qquad v|_\gamma = 0.$$

In virtue of Eqs. (2.7), we have on the characteristics

$$\frac{dv}{dt} + a_0(t)v = 0, \qquad v|_{t=0} = 0.$$

By the theorem of existence and uniqueness of the solution of a Cauchy problem for ordinary differential equations, we have $v(t) = 0$. Hence, $u_1(\vec{x}) = u_2(\vec{x})$, and the solution of the Cauchy problem (2.1), (2.2) is unique. This proves the theorem.

From this proof it follows that to solve the Cauchy problem (2.1), (2.2), it suffices

(1) to construct the characteristics of the system of equations (2.3), which pass through the surface γ, and find $\vec{x} = \vec{X}(t, s_1, \ldots, s_{n-1})$, the solution of the Cauchy problem (2.3), (2.4);

(2) to solve the family of Cauchy problems (2.7), i.e., to find $u = U(t, s_1, \ldots, s_{n-1})$;

(3) to find the solution of Eqs. (2.5)

$$t = T(\vec{x}), \qquad s_k = S_k(\vec{x}), \qquad k = \overline{1, n-1}; \tag{2.8}$$

(4) to calculate, using (2.8),

$$u(\vec{x}) = U\big(T(\vec{x}), S_1(\vec{x}), \ldots, S_{n-1}(\vec{x})\big).$$

Example 2.1. Making use of the above scheme, find the solution of the Cauchy problem

$$x\frac{\partial u}{\partial x} - yz\frac{\partial u}{\partial z} = 0, \qquad u\Big|_{z=1} = x^y.$$

Solution. 1. In this case, the characteristic system of equations has the form

$$\dot{x} = x, \qquad \dot{y} = 0, \qquad \dot{z} = -yz. \tag{2.9}$$

The equation of the Cauchy surface, $z = 1$, will be written in parametric form as

$$x = s_1, \quad y = s_2, \quad z = 1, \quad (s_1, s_2) \in \mathbb{R}^2,$$

and the initial conditions will be

$$x\Big|_{t=0} = s_1, \quad y\Big|_{t=0} = s_2, \quad z\Big|_{t=0} = 1.$$

From (2.9) we find

$$x = X(t, s_1, s_2) = s_1 e^t, \quad y = Y(t, s_1, s_2) = s_2,$$
$$z = Z(t, s_1, s_2) = e^{-s_2 t}.$$

2. The Cauchy problem (2.7) takes the form

$$\dot{u} = 0, \qquad u\Big|_{t=0} = s_1^{s_2},$$

whence

$$u = V(t, s_1, s_2) = s_1^{s_2}.$$

3. Solve the system of equations

$$x = s_1 e^t, \qquad y = s_2, \qquad z = e^{s_2 t}$$

with respect to t, s_1, and s_2 to get

$$s_2 = y, \qquad s_1 = e^{-t}x, \qquad s_2 t = -\ln z,$$

whence

$$s_1 = S_1(x, y, z) = xe^{(\ln z)/y}, \quad s_2 = S_2(x, y, z) = y,$$
$$t = T(x, y, z) = -\frac{\ln z}{y}.$$

4. Finally, we obtain

$$u = U(T(x, y, z), S_1(x, y, z), S_2(x, y, z)) = \left[xe^{(\ln z)/y}\right]^y = zx^y,$$

which coincides with (1.18).

CHAPTER 2

Reduction of second order equations to the canonical form

3 Classification of second order equations

♦ An equation relating independent variables x and y, an unknown function u and its partial derivatives of up to the second order inclusive is called *a second order partial differential equation*

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \tag{3.1}$$

♦ Equation (3.1) is said to be *linear with respect to higher derivatives* if it can be represented in the form

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + F(x, y, u, u_x, u_y) = 0, \tag{3.2}$$

where $a_{ik} = a_{ik}(x, y)$, $i, k = 1, 2$.

♢ If $a_{ik} = a_{ik}(x, y, u)$, the equation is called *quasilinear*.

♦ A second order partial differential equation is called *linear* if it is linear both with respect to the higher derivatives and with respect to the function u itself and to its first derivatives:

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y + cu + f = 0. \tag{3.3}$$

If $f \equiv 0$, Eq. (3.3) is called *homogeneous*.

Let us consider Eq. (3.2) which is linear with respect to the higher derivatives. Let us perform the change of variables

$$\alpha = \varphi(x, y), \qquad \beta = \psi(x, y). \tag{3.4}$$

♢ We would like to choose α and β so as to make Eq. (3.2) in new variables as simple as possible.

Let us find expressions for the derivatives of the function u with respect to the new variables:

$$\begin{aligned} &u_x = \alpha_x u_\alpha + \beta_x u_\beta, \qquad u_y = \alpha_y u_\alpha + \beta_y u_\beta;\\ &u_{xx} = \alpha_x^2 u_{\alpha\alpha} + 2\alpha_x\beta_x u_{\alpha\beta} + \beta_x^2 u_{\beta\beta} + u_\alpha\alpha_{xx} + u_\beta\beta_{xx};\\ &u_{xy} = \alpha_x\alpha_y u_{\alpha\alpha} + (\alpha_x\beta_y + \alpha_y\beta_x)u_{\alpha\beta} +\\ &\qquad + \beta_y\beta_x u_{\beta\beta} + u_\alpha\alpha_{xy} + u_\beta\beta_{xy};\\ &u_{yy} = \alpha_y^2 u_{\alpha\alpha} + 2\alpha_y\beta_y u_{\alpha\beta} + \beta_y^2 u_{\beta\beta} + u_\alpha\alpha_{yy} + u_\beta\beta_{yy}. \end{aligned} \tag{3.5}$$

Substitute these expressions into Eq. (3.2) to get

$$\bar a_{11}u_{\alpha\alpha} + 2\bar a_{12}u_{\alpha\beta} + \bar a_{22}u_{\beta\beta} + \bar F(\alpha, \beta, u, u_\alpha, u_\beta) = 0, \tag{3.6}$$

where

$$\begin{aligned} \bar a_{11} &= a_{11}\alpha_x^2 + 2a_{12}\alpha_x\alpha_y + a_{22}\alpha_y^2,\\ \bar a_{12} &= a_{11}\alpha_x\beta_x + a_{12}(\alpha_x\beta_y + \alpha_y\beta_x) + a_{22}\alpha_y\beta_y,\\ \bar a_{22} &= a_{11}\beta_x^2 + 2a_{12}\beta_x\beta_y + a_{22}\beta_y^2. \end{aligned} \tag{3.7}$$

Let us assume, for instance, that $\bar a_{11} = 0$ (or $\bar a_{22} = 0$). Then, to determine the functions $\alpha(x, y)$ and $\beta(x, y)$, it is necessary to solve the following first order partial differential equation:

$$a_{11}z_x^2 + 2a_{12}z_yz_x + a_{22}z_y^2 = 0. \tag{3.8}$$

♦ Equation (3.8) is called *the characteristic equation for the quasilinear second order equation* (3.2), and the curve defined by $z(x,y) = C$, where $z = z(x,y)$ is a continuously differentiable solution of (3.8), is called *a characteristic line*, or *a characteristic*, of Eq. (3.2).

Lemma 3.1. *If $z = z(x,y)$ is a particular solution of Eq.* (3.8), *satisfying the condition*

$$\frac{\partial z}{\partial y} \neq 0,$$

then $z(x,y) = C$ is the total integral of the equation

$$a_{11}dy^2 - 2a_{12}dx\,dy + a_{22}dx^2 = 0. \tag{3.9}$$

Proof. Let $z(x,y)$ be a solution of Eq. (3.8) and the equation $z(x,y) = C$ be solvable for y, that is $y = f(x,C)$. Then, by the rule of differentiation for implicitly specified functions

$$\frac{dy}{dx} = -\frac{z_x}{z_y}\Big|_{y=f(x,C)}.$$

Substitute this derivative into Eq. (3.9) to get

$$a_{11}\Big(\frac{dy}{dx}\Big)^2 - 2a_{12}\frac{dy}{dx} + a_{22} = \Big[a_{11}\Big(-\frac{z_x}{z_y}\Big)^2 - 2a_{12}\Big(-\frac{z_x}{z_y}\Big) + a_{22}\Big]\Big|_{y=f(x,C)} = 0.$$

Since Eq. (3.8) holds for all x and y in the region where a solution exists, this proves the lemma.

The converse lemma is also valid.

Lemma 3.2. *If $\varphi(x,y) = C$ is the total integral of Eq.* (3.9), *the function $z = \varphi(x,y)$ is a particular solution of Eq.* (3.8).

Proof is similar.

Solving Eq. (3.9) for dy/dx, we see that this equation can be subdivided into two equations

$$\frac{dy}{dx} = \frac{a_{12} + \sqrt{D}}{a_{11}}, \quad \frac{dy}{dx} = \frac{a_{12} - \sqrt{D}}{a_{11}}, \quad D = a_{12}^2 - a_{11}a_{22}, \tag{3.10}$$

which are called the differential equations of characteristics for (3.2).

By virtue of the lemmas proved above, the total integrals of Eqs. (3.10), $\varphi(x,y) = C_1$ and $\psi(x,y) = C_2$, specify two families of characteristics for Eq. (3.2).

♢ Direct check yields

$$\bar{D} = \bar{a}_{12}^2 - \bar{a}_{11}\bar{a}_{22} = DJ^2,$$

where

$$J = \alpha_x\beta_y - \beta_x\alpha_y = \frac{\partial(\alpha,\beta)}{\partial(x,y)} \neq 0$$

(J is the Jacobian of the passage to new coordinates). Hence, the change of variables (3.4) does not change the sign of D.

♦ Equation (3.2) for a point M is called

(1) *hyperbolic* if at this point $D > 0$;

(2) *elliptic* if at this point $D < 0$;

(3) *parabolic* if at this point $D = 0$.

♦ Equation (3.2) is called *hyperbolic* (*elliptic, parabolic*) in a region G if it belongs to the hyperbolic (elliptic, parabolic) type at each point of the region G.

◊ One and the same equation may belong to different types at different points of the region of its definition (see Example 4.2).

4 The canonical form of a second order differential equation with two independent variables

◊ Consider a region G at every point of which an equation of the form (3.2) is of the same type. Then from (3.10) it follows that two characteristics pass through each point of the region G with the characteristics being respectively real and different for the hyperbolic type, complex and different for the elliptic type, and real and coinciding for the parabolic type.

1. Hyperbolic equations ($D > 0$)

The total integrals

$$\varphi(x, y) = C_1 \quad \text{and} \quad \psi(x, y) = C_2$$

define a real family of characteristics. Let us choose new variables as follows:

$$\alpha = \varphi(x, y), \qquad \beta = \psi(x, y).$$

Then $\bar{a}_{11} = \bar{a}_{22} = 0$ and $\bar{D} = \bar{a}_{12}^2 = DJ^2 > 0$. Divide the left and the right side of Eq. (3.6) by $\bar{a}_{12} \neq 0$ to get

$$u_{\alpha\beta} + \bar{F}(\alpha, \beta, u, u_\alpha, u_\beta) = 0. \tag{4.1}$$

♦ Equation (4.1) is called *a hyperbolic equation in the first canonical form.*

Perform in (4.1) the change

$$\beta = \frac{\xi + \eta}{2}, \qquad \alpha = \frac{\xi - \eta}{2}.$$

Then

$$u_\alpha = \frac{u_\xi + u_\eta}{2}, \quad u_\beta = \frac{u_\xi - u_\eta}{2}, \quad u_{\alpha\beta} = \frac{u_{\xi\xi} - u_{\eta\eta}}{4}.$$

Thus, Eq. (4.1) takes the form

$$u_{\xi\xi} - u_{\eta\eta} + F(\xi, \eta, u, u_\xi, u_\eta) = 0. \tag{4.2}$$

♦ Equation (4.2) is called *a hyperbolic equation in the second canonical form.*

2. Elliptic equations ($D < 0$)

Equations (3.9) have two complex conjugate total integrals. Put

$$\alpha = \varphi(x, y), \qquad \beta = \varphi^*(x, y),$$

which gives

$$u_{\alpha\beta} + \bar{F} = 0.$$

Introducing the real variables

$$\beta = \frac{\xi + i\eta}{2}, \qquad \alpha = \frac{\xi - i\eta}{2}$$

we obtain

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + F(\xi, \eta, u, u_\xi, u_\eta) = 0. \tag{4.3}$$

♦ Equation (4.3) is called *an elliptic equation in canonical form.*

3. Parabolic equations ($D = 0$)

Equations (3.10) coincide, and there exists only one total integral: $\varphi(x, y) = C$. Put

$$\alpha = \varphi(x, y), \qquad \beta = \psi(x, y),$$

where $\psi(x, y)$ is an arbitrary function independent of $\varphi(x, y)$. By definition $\bar{a}_{11} = 0$, and the condition $D = 0$ implies

$$\bar{D} = \bar{a}_{12}^2 - \bar{a}_{11}\bar{a}_{22} = 0,$$

whence

$$\bar{a}_{12} = \sqrt{\bar{a}_{11}\bar{a}_{22}} = 0.$$

Thus, we have

$$u_{\beta\beta} + F(\alpha, \beta, u_\alpha, u_\beta) = 0. \tag{4.4}$$

♦ Equation (4.4) is called *a parabolic equation in canonical form.*

◊ If F is independent of u_α, then Eq. (4.4) is an ordinary differential equation which depends on α as of a parameter.

◊ In the case of many variables, the classification of equations is not so simple. Equations of the form

$$\sum_{i,j=1}^{n} g_{ij}(\vec{x})\frac{\partial^2 u}{\partial x_i \partial x_j} + F(\vec{x}, u, \nabla u) = 0, \quad \nabla = \frac{\partial}{\partial \vec{x}}, \tag{4.5}$$

are called *quasilinear second order equations with n variables.*

Generally speaking, for $n \geqslant 3$ there is no change of variables which would reduce the coefficients g_{ij} to the diagonal form in the whole space $\mathbb{R}^n_x$ (i.e., $g_{ij}(\vec{x}) = g_i(\vec{x})\delta_{ij}$, where δ_{ij} is Kronecker's symbol). However, the reduction to the diagonal form is possible at any preassigned point of the range of the variable $\vec{x}$. In this case, the number of positive, negative, and zero coefficients $g_i(\vec{x})$ does not depend on the way of reduction (due to the law of inertia for a quadratic form). This can serve as a basis for the classification of Eq. (4.5) for different regions.

Below we will come across three types of equation:

(1) elliptic

$$\sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} + F(\vec{x}, u, \nabla u) = 0;$$

(2) hyperbolic

$$\frac{\partial^2 u}{\partial x_1^2} - \sum_{k=2}^{n} \frac{\partial^2 u}{\partial x_k^2} + F(\vec{x}, u, \nabla u) = 0;$$

(3) parabolic

$$\sum_{k=2}^{n} \frac{\partial^2 u}{\partial x_k^2} + F(\vec{x}, u, \nabla u) = 0.$$

In conclusion, we formulate **a scheme for reducing Eq. (3.2) to the canonical form:**

✓ determine the coefficients a_{11}, a_{12}, and a_{22} in accordance with the form of Eq. (3.2);

✓ determine the regions of constant sign for the discriminant $D = a_{12}^2 - a_{11}a_{22}$ where Eq. (3.2) retains its type and find the type of the given equation in each of these regions;

✓ write down the characteristic equation (3.9) for the coefficients of the original equation $a_{11}dy^2 - 2a_{12}dy\,dx + a_{22}dx^2 = 0$ and find its first integrals $\varphi(x, y) = C_1$ and $\psi(x, y) = C_2$;

✓ write down formulas for passing from the original variables (x, y) to new ones ($\alpha = \varphi(x, y)$, $\beta = \psi(x, y)$) proceeding from the solutions of the characteristic equation taking into account its type features;

✓ express the derivatives with respect to the original variables in terms of the derivatives with respect to the new variables, in accordance with (3.5), and substitute them into the original equation;

✓ using (4.4), express the original variables in terms of the new ones, that is, find $x = \bar{\varphi}(\alpha, \beta)$ and $y = \bar{\psi}(\alpha, \beta)$;

✓ in the expression obtained, eliminate the original variables by making use of (4.4) and collect similar terms thereby reducing the original equation to one of the canonical forms.

The above simplifying transformations of the original equation result in a canonical form which allows one to find, in particular cases, a general solution. Particular methods for finding a general solution are considered below for specific examples.

Example 4.1. Find the general solution of the equation

$$3u_{xx} + 14u_{xy} + 8u_{yy} = 0. \tag{4.6}$$

Solution. 1. Compose a characteristic equation

$$3dy^2 - 14dy\,dx + 8dx^2 = 0$$

or

$$3\Big(\frac{dy}{dx}\Big)^2 - 14\frac{dy}{dx} + 8 = 0.$$

Solving this equation for dy/dx, we obtain

$$\frac{dy}{dx} = \frac{14 \pm \sqrt{D}}{6}, \qquad D = 196 - 4 \cdot 3 \cdot 8 = 100 > 0.$$

Thus, Eq. (4.6) is of hyperbolic type all over the plane. Integrate the obtained characteristic equations to get

$$\frac{dy}{dx} = 4, \qquad \frac{dy}{dx} = \frac{2}{3}, \qquad C_1 = y - 4x, \qquad C_2 = y - \frac{2}{3}x.$$

2. Perform the change of variables

$$\alpha = y - 4x, \qquad \beta = y - \frac{2}{3}x$$

to get

$$\alpha_x = -4, \qquad \alpha_y = 1, \qquad \alpha_{xx} = \alpha_{xy} = \alpha_{yy} = 0,$$
$$\beta_x = -\frac{2}{3}, \qquad \beta_y = 1, \qquad \beta_{xx} = \beta_{xy} = \beta_{yy} = 0.$$

Then

$$\begin{aligned} u_x &= u_\alpha \alpha_x + u_\beta \beta_x = -4u_\alpha - \frac{2}{3}u_\beta, \\ u_y &= u_\alpha \alpha_y + u_\beta \beta_y = u_\alpha + u_\beta, \\ u_{xx} &= u_{\alpha x}\alpha_x + u_\alpha \alpha_{xx} + u_{\beta x}\beta_x + u_\beta \beta_{xx} = \\ &= \alpha_x(u_{\alpha\alpha}\alpha_x + u_{\alpha\beta}\beta_x) + u_\alpha \alpha_{xx} + \beta_x(u_{\beta\alpha}\alpha_x + u_{\beta\beta}\beta_x) + u_\beta \beta_{xx} = \\ &= 16u_{\alpha\alpha} + \frac{16}{3}u_{\alpha\beta} + \frac{4}{3}u_{\beta\beta}. \end{aligned}$$

Similarly,

$$u_{xy} = -4u_{\alpha\alpha} - \frac{14}{3}u_{\alpha\beta} - \frac{2}{3}u_{\beta\beta},$$
$$u_{yy} = u_{\alpha\alpha} + 2u_{\alpha\beta} + u_{\beta\beta}.$$

3. Substitute the obtained expressions for partial derivatives into the original equation to obtain

$$-\frac{100}{3}u_{\beta\alpha} = 0$$

or

$$u_{\beta\alpha} = 0.$$

Hence the general solution of Eq. (4.6) has the form

$$u(\alpha, \beta) = p(\alpha) + q(\beta),$$

where $p(\alpha)$ and $q(\beta)$ are arbitrary functions. Returning to the original variables, we can write

$$u(x, y) = p(y - 4x) + q(y - 2x/3).$$

Example 4.2. Reduce to a canonical form and find the general solution of the equation

$$\operatorname{sign} y\, u_{xx} + 2u_{xy} + u_{yy} = 0, \quad \operatorname{sign} y = \begin{cases} 1, & y > 0; \\ 0, & y = 0; \\ -1, & y < 0. \end{cases}$$

Solution. Since $a_{11} = \operatorname{sign} y$, $a_{12} = 1$, $a_{22} = 1$, then

$$D = a_{12}^2 - a_{11}a_{22} = 1 - \operatorname{sign} y = \begin{cases} 0, & y > 0; \\ 1, & y = 0; \\ 2, & y < 0, \end{cases}$$

and the original equation is parabolic in the upper half-plane ($y > 0$) and hyperbolic in the remaining region ($y \leqslant 0$). By virtue of this, we have to consider these regions separately. Cases of hyperbolic equations are considered like in Example 1. Consider the case of a parabolic equation ($y > 0$, $\operatorname{sign} y = 1$, $a_{11} = 1$, $\Delta = 0$).

1) *Reduction to a canonical form*

According to (3.10), we have one characteristic equation with a total integral $C = y - x$ that defines one of the two new variables, e.g., $\alpha = y - x$. Then for the other new variable β we can take any function of x, y, which is independent of $\alpha = y - x$. For simplicity, we choose $\beta = x$. Then, according to (3.5), we obtain a canonical form of the equation in the region of parabolicity

$$u_{\beta\beta} = 0.$$

2) *Finding the general solution*

Make use of the substitution $v = u_\beta$ to reduce the order of the equation. Then we have an equation $v_\beta = 0$ whose solution is an arbitrary function of α, hence $v = p(\alpha)$. Integrating the last equation with respect to the variable β, we obtain

$$u(\alpha, \beta) = p(\alpha)\beta + q(\alpha),$$

where $q(\alpha)$ is another arbitrary function of the variable α. Returning to the original variables, we obtain the general solution in the form

$$u(x, y) = xp(y - x) + q(y - x).$$

Note that in the region of parabolicity there is only one characteristic equation. Therefore, only one variables can be determined uniquely, while the other can be chosen arbitrarily. This arbitrariness gives rise to a set of solutions $u(x, y)$. For instance, the choice $\alpha = y - x$, $\beta = y + x$ results in a general solution of the form

$$u(x, y) = (x + y)p(y - x) + q(y - x).$$

Example 4.3. Find the general solution of the equation

$$u_{xy} + u_x + u_y + u = 0.$$

Solution. Since the equation is written in canonical form, the problem is reduced to finding a general solution. The order of the equation can be reduced by introducing a new function $v = u_y + u$. Then the original equation will take the form

$$v_x + v = 0.$$

Its solution is written as

$$v(x, y) = \varphi(y)e^{-x},$$

where $\varphi(y)$ is an arbitrary function of y. Returning to the function $u(x, y)$, we have

$$u_y + u = e^{-x}\varphi(y), \tag{4.7}$$

that is a linear equation in u, u_y. By analogy with the Bernoulli method for ordinary differential equations, we seek the solution of Eq. (4.7) in the form

$$u(x, y) = a(x, y)b(x, y), \tag{4.8}$$

where $a(x, y)$ and $b(x, y)$ are some functions one of which can be taken arbitrarily. Substitution of (4.8) into (4.7) yields

$$a_y b + a b_y + ab = e^{-x}\varphi(y).$$

Putting

$$a_y + a = 0,$$

we obtain an equation for the function $b(x, y)$:

$$a b_y = e^{-x}\varphi(y).$$

From the first equation we have

$$a = e^{-y},$$

and from the second one

$$b_y = e^{y-x}\varphi(y).$$

Integration yields

$$b(x, y) = \int e^{y-x}\varphi(y)dy + g(x).$$

Here $g(x)$ is an arbitrary function. Returning to the function $u(x, y)$, we find

$$u(x, y) = a(x, y)b(x, y) = e^{-y}\Big[e^{-x}\int e^{y}\varphi(y)dy + g(x)\Big]$$

or

$$u(x, y) = e^{-x}q(y) + e^{-y}g(x).$$

Here, instead of $\varphi(y)$, we have introduced a new arbitrary function

$$q(y) = e^{-y}\int e^{y}\varphi(y)dy.$$

Thus, the general solution has the form

$$u(x,y) = e^{-x}q(y) + e^{-y}g(x),$$

where $q(y)$ and $g(x)$ are arbitrary functions of the variables y and x, respectively.

Example 4.4. Find the solution of the Cauchy problem

$$u_{xy} + u_x + u_y + u = 0,$$
$$u(x,y)\big|_{y=3x} = 0, \qquad u_y(x,y)\big|_{y=3x} = e^{-4x}.$$

Solution. The general solution of the equation was obtained in Example 4.3. It has the form

$$u(x,u) = e^{-x}q(y) + e^{-y}g(x). \tag{4.9}$$

From the initial conditions we find the functions $p(x)$ and $q(x)$. To begin with we calculate u_y:

$$u_y(x,u) = e^{-x}q'(y) - e^{-y}g(x).$$

Then, putting $y = 3x$ and substituting u and u_y into the initial conditions, we obtain a system of equations to find the functions $g(y)$ and $q(x)$:

$$u(x,y)\big|_{y=3x} = e^{-x}q(3x) + e^{-3x}g(x) = 0, \tag{4.10}$$
$$u_y(x,y)\big|_{y=3x} = e^{-x}q'(3x) - e^{-3x}g(x) = e^{-4x}. \tag{4.11}$$

Here the prime denotes the derivative with respect to the argument $3x$, i.e.,

$$q'(3x) = \frac{dq(3x)}{d(3x)}.$$

Combining Eq. (4.10) and Eq. (4.11), we have

$$e^{-x}[q(3x) + q'(3x)] = e^{-4x},$$

whence

$$q(3x) = e^{-3x}(3x + C),$$

where C is an arbitrary constant. Correspondingly,

$$q(z) = (z + C)e^{-z}. \tag{4.12}$$

Substitution of (4.12) into (4.9) yields

$$g(x) = -e^{-x}[3x + C]. \tag{4.13}$$

Returning to $u(x,y)$, in view of (4.13), we obtain

$$u(x,y) = e^{-(x+y)}(y + C) - e^{-y}e^{-x}(3x + C) = e^{-(x+y)}(y + C - 3x - C).$$

Thus, the solution of the Cauchy problem has the form

$$u(x,y) = e^{-(x+y)}(y - 3x).$$

CHAPTER 3

Partial Differential Equations in Physical Problems

5 The linear chain

Let us consider a transition from the mechanics of a system of material points to the distributed mass mechanics.

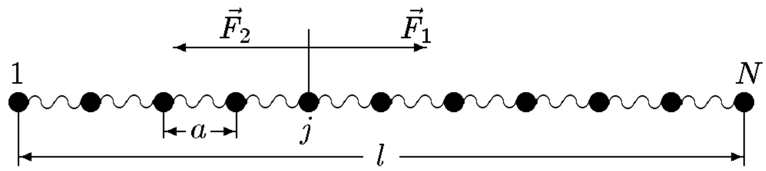

Fig. 27.

As an example we imagine a linear chain consisting of N identical material points of mass m each, connected by springs. Let these springs have the same coefficient of rigidity k. Assume that in equilibrium the chain has a length l and the distances between any two neighboring points are the same and equal to a. For the case where equilibrium is violated, only one-dimensional motions along the segment are admitted. The boundary points are assumed to be immobile. The jth point is subject to the action of counterdirected elastic forces $\vec{F}_1$ and $\vec{F}_2$ whose moduli, according to the Hooke law, are given by (see Fig. 27) $F_1 = k(u_{j+1} - u_j)$ and $F_2 = k(u_j - u_{j-1})$, where u_j is the displacement of the jth point. As a result Newton's second law for the jth point has the form

$$m\ddot{u}_j = k(u_{j+1} - u_j) - k(u_j - u_{j-1}), \quad j = \overline{2, N-1}, \qquad (5.1)$$
$$u_1 = u_N = 0.$$

To pass to a continuous mass distribution over the chain, we consider the limit of $a \to 0$, $m \to 0$ subject to the condition that the linear density $\rho = m/a$ and Young's modulus $E = ka$ remain finite. Since the chain dimensions remain unchanged, the number of points N will tend to infinity. In this limit, the number of the jth point is replaced by a continuous quantity x which specifies the position of the point on the segment, that is, $j \to x$. The distance a between two neighboring points is replaced by a differential: $a \to dx$. As a result the displacement $u_j(t)$ becomes a function of two variables: x and t. Thus, we have

$$u_j - u_{j-1} \to a\frac{\partial u}{\partial x},$$
$$(u_{j+1} - u_j) - (u_j - u_{j-1}) \to a^2\frac{\partial^2 u}{\partial x^2}.$$

In the limit, the system of equations (5.1) transforms into the partial differential equation for the function $u(x,t)$

$$\rho\frac{\partial^2 u}{\partial t^2} = E\frac{\partial^2 u}{\partial x^2}, \qquad u|_{x=0} = u|_{x=l} = 0. \qquad (5.2)$$

This equation describes longitudinal waves propagating in a rod with velocity $\sqrt{E/\rho}$ (see for details the section devoted to longitudinal oscillations of an elastic rod).

6 The equations for a vibrating string

Consider a solid body whose longitudinal dimensions are much larger than the transverse ones ($l \gg a$, where l is the length and a the maximum transverse dimension of the body). Thus, we arrive at the notion of string as an ideal one-dimensional object. If the tension acting on this body is much greater than the bending resistance, the latter can be neglected. This gives rise to the notion of an ideally flexible string.

Assume that the position of the string at rest coincides with the Ox-axis. Suppose that the string moves under the action of transverse forces in one plane xOu, where by u is meant the displacement of the string from the equilibrium position at a point with coordinate x at time t. Then the relation $u = u(x,t)$ determines the string profile in the plane xOu at time t. Our task is to construct an equation which would be satisfied by the function $u = u(x,t)$. Assume that the string is elastic, i.e., obeys the Hooke law: the change in tension is proportional to the change in length of the string.

Denote the linear density of the string and the linear density of the external transverse forces by $\rho(x)$ and $F(x,t)$, respectively. Since the string does not offer resistance to bending, its tension $\vec{T}(x,t)$ at any coordinate x and time t is directed along the tangent to the string at the point x (see Fig. 28).

The length of an element of the string in equilibrium dx and the length of the flexed portion of the string disturbed from an equilibrium condition dl are well known to be related as

$$dl = \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}\,dx.$$

Assuming that the quantity

$$\frac{\partial u}{\partial x} = \operatorname{tg}\alpha(x)$$

is small and neglecting the higher order quantities, we have $dl \approx dx$. This means that under the assumptions made the string length can be considered invariable in vibrations since the elongation is negligible (infinitesimal of higher order) compared to the initial length, and we neglect this quantity. From here it immediately follows, according to the Hooke law, that the tension vector $\vec{T}(x,t)$ may vary only its direction (in a tangent), while its modulus $|\vec{T}(x,t)|$ will be a constant independent of x and t, that is, $|\vec{T}(x,t)| = T_0 = \text{const}$.

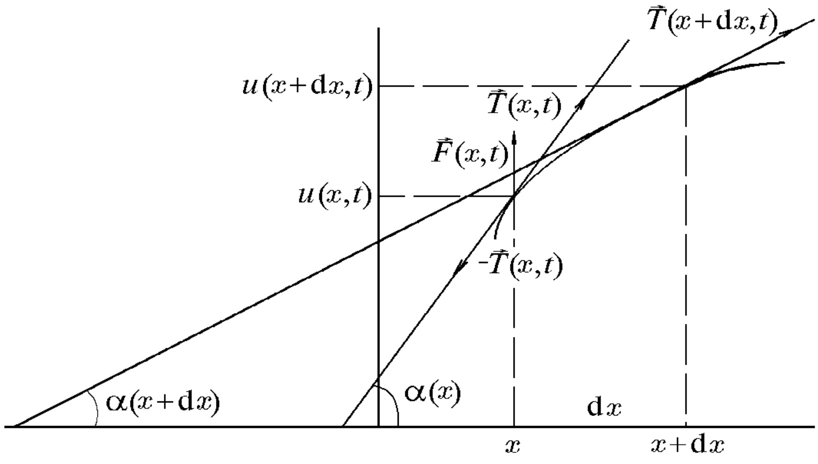

Fig. 28.

Note that under the assumptions made the relationships

$$\begin{aligned} \sin\alpha &= \frac{\operatorname{tg}\alpha}{\sqrt{1+\operatorname{tg}^2\alpha}} = \frac{\partial u}{\partial x}\Big/\sqrt{1+\Big(\frac{\partial u}{\partial x}\Big)^2} \approx \frac{\partial u}{\partial x},\\ \cos\alpha &= \frac{1}{\sqrt{1+\operatorname{tg}^2\alpha}} = 1\Big/\sqrt{1+\Big(\frac{\partial u}{\partial x}\Big)^2} \approx 1 \end{aligned} \tag{6.1}$$

are valid. Let us consider the forces acting on a string segment $(x, x+dx)$. These are the forces due to tension $\vec{T}(x+dx,t)-\vec{T}(x,t)$ and the external forces $\vec{F}(x,t)dx$ (the density $\vec{F}(x,t)$ can be considered invariable on the interval dx (see Fig. 28)) whose sum, according to Newton's second law, determines the acceleration $\partial^2 u(x,t)/\partial t^2$ of the string segment whose mass is $dm=\rho(x)dx$. The projection of this vector equality on the Ox axis, in view of (6.1), yields

$$T_0\cos(\alpha(x+dx)) - T_0\cos(\alpha(x)) \simeq T_0 - T_0 = 0,$$

while its projection on the Ou-axis results in

$$T_0\sin(\alpha(x+dx)) - T_0\sin(\alpha(x)) + F(x,t)dx = \rho(x)\frac{\partial^2 u(x,t)}{\partial t^2}dx,$$

from which, in view of the relationship

$$\begin{gathered} \frac{\sin(\alpha(x+dx)) - \sin(\alpha(x))}{dx} \approx \\ \approx \frac{\partial u(x+dx,t)/\partial x - \partial u(x,t)/\partial x}{dx} = \frac{\partial^2 u(x,t)}{\partial x^2} \end{gathered}$$

we obtain the equation

$$\rho(x)\frac{\partial^2 u(x,t)}{\partial t^2} = T_0\frac{\partial^2 u(x,t)}{\partial x^2} + F(x,t), \tag{6.2}$$

which is called the *equation of small transverse vibrations of string*. Such vibrations are called forced for $F(x,t)\not\equiv 0$ and free for $F(x,t)\equiv 0$.

If the string density is a constant, denoting

$$a^2 = \frac{T_0}{\rho}, \qquad f = \frac{F}{\rho},$$

we may write Eq. (6.2) in the form

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + f, \tag{6.3}$$

that was considered by Euler and d'Alembert as early as the 18th century. Later on we will see that equations of the form (6.3) arise in solving other physical problems. Therefore, irrespective of the physical sense of the quantity $u(x,t)$, we shall also refer to Eq. (6.3) as a one-dimensional wave equation.

Returning to Eq. (6.2), it should be noted that, as this is a second order partial differential equation, its solution is not unique since generally contains two arbitrary functions. To uniquely describe the process of vibration of a string, Eq. (6.2) is complemented with some conditions following from the physical statement of the problem. As a rule, the following conditions are imposed on the equation of a vibrating string (6.2).

1. *The Cauchy problem*

If the behavior of the boundary points of the string is not specified in advance from physical considerations (e.g., the string is infinitely long), unique solution can be obtained under the initial conditions

$$u(x,t)|_{t=0} = \varphi(x), \quad \frac{\partial u(x,t)}{\partial t}\Big|_{t=0} = \psi(x), \tag{6.4}$$

specifying the displacements and velocities (momenta) of any point of the string at the time zero.

♦ The problem of finding a solution to Eq. (6.2), satisfying initial conditions (6.4), is *a Cauchy problem.*

2. *The mixed problem*

If the behavior of the boundary points of the string (e.g., a and b, so that the string length $l = b-a$) is specified beforehand, the Cauchy problem should be complemented with boundary conditions. Let us consider three main (classical) kinds of boundary conditions.

(a) **the first kind boundary conditions**: the boundary point of the string moves in accordance with some law, i.e., $u(x,t)|_{x=a} = \mu(t)$; naturally, $\mu(t) = 0$ corresponds to the case the point a is rigidly fixed (i.e., the point a is immobile);

(b) **the second kind boundary conditions**: a given force $\nu(t)$ acts on the boundary point; then

$$T_0 \sin\alpha|_{x=a} \simeq T_0 \partial u/\partial x|_{x=a} = \nu(t),$$

from which we have

$$\frac{\partial u}{\partial x}\Big|_{x=a} = \frac{\nu(t)}{T_0},$$

i.e., for $\nu(t) = 0$ the end $x = a$ of the string moves freely (is not fixed);

(c) **the third kind boundary conditions**: the boundary point of the string is fixed elastically with the coefficient of rigidity of fixing h; then, in accordance with the Hooke law,

$$\Big(T_0\frac{\partial u}{\partial x} + hu\Big)\Big|_{x=a} = 0.$$

Similarly, the behavior of the second boundary point $x = b$ is specified. If $x = b \to \infty$, we have a problem for a semi-infinite string $]a, \infty[$ with a single boundary condition.

♦ The problem of finding a solution to Eq. (6.2) subject to the initial conditions (6.4) and any one of the boundary conditions (a), (b), and (c) is called *a mixed problem.*

♢ The above classification of boundary conditions is classical and exhaustive for the majority of physical problems. Similar boundary conditions arise in other physical problems leading to a one-dimensional wave equation. This is why the boundary conditions (a), (b), and (c), irrespective of the physical sense, are called boundary conditions of the first, second, and third kind, respectively.

Boundary conditions which can not be put into the classical scheme can be obtained by selecting segments adjoining the ends of a rod, $]a, a+dx[$ or $]b-dx, b[$, and writing the equation of motion for these segments, as was done in deriving the equation.

3. *The boundary value problem*

The influence of the boundary conditions on the vibration of the string may fall off with time (e.g., due to the resistive forces), and the vibratory process will be practically determined, beginning from some point in time, by the boundary conditions. Thus, at long times the initial conditions (6.4) become inessential, and we arrive at a purely boundary value problem of finding a solution to Eq. (6.3) for $a \leqslant x \leqslant b$ as $t \to \infty$. The boundary value problems, depending on the boundary conditions (a), (b), and (c), are called *the first*, *the second*, and *the third boundary value problem*.

In this statement, we may distinguish three important types of problems: seeking for an equilibrium equation for a string subject to the action of stationary external forces and problems on natural and forced vibrations of a string. In the first case, the problem reduces to finding a solution to the ordinary differential equation

$$a^2\frac{d^2u(x)}{dx^2} + f(x) = 0.$$

If we seek a solution to the problem for a freely vibrating string as a periodic function

$$u(x,t) = v(x)\sin\omega t \qquad \text{or} \qquad u(x,t) = v(x)\cos\omega t,$$

we obtain for the amplitude $v(x)$ the equation

$$\frac{d^2v(x)}{dx^2} + \frac{\omega^2}{a^2}v(x) = 0, \tag{6.5}$$

called an equation of natural vibrations of a string, which can be solved uniquely subject to boundary conditions only. The problem of seeking non-trivial solutions to Eq. (6.5), which would satisfy given boundary conditions, is a special case of the boundary value problem. It can easily be noticed that from the mathematical viewpoint the solution of this problem reduces to the solution of an eigenvalue (natural frequencies ω_n) and eigenfunction problem (see Sec. "The Sturm–Liouville problem") to determine the amplitudes $v_n(x)$.

For a forced-vibrating string with the generating force $f(x,t)$ being periodic with frequency ν and amplitude $a^2f(x)$, i.e.,

$$f(x,t) = a^2f(x)\sin\nu t, \quad f(x,t) = a^2f(x)\cos\nu t, \tag{6.6}$$

the solution can be sought for as a function with unknown amplitude $v(x)$ and frequency equal to the frequency of the generating force, i.e., as

$$u(x,t) = v(x)\sin\nu t \quad \text{or} \quad u(x,t) = v(x)\cos\nu t. \tag{6.7}$$

Substituting (6.6) and (6.7) into Eq. (6.3), we obtain for the function $v(x)$ the stationary equation

$$\frac{d^2v(x)}{dx^2} + \frac{\nu^2}{a^2}v(x) = -f(x), \tag{6.8}$$

describing the forced vibrations of string, which is, as we will see below, a one-dimensional analog to the Helmholtz equation. Naturally, a unique solution to Eq. (6.8) is determined only by the boundary conditions; however, the character of the solution depends substantially on the proportion between the frequency ν of the generating force and the natural frequencies ω_n of the string. When ν coincides with one of the natural frequencies ω_n, the resonance effect well known from general physics occurs.

In conclusion it should be noted that the specific external forces (the resistance of the environment, the gravity force, etc.) can be taken into account in a natural manner in deriving the equation of vibrations completely complying with the physical statement of the problem. On the other hand, the difficulties involved in constructing smooth solutions of differential equations with piecewise continuous distributions of physical characteristics ($\rho(x)$, $F(x)$, and the like) can be overcome on the basis of the notion of generalized solutions in the class of generalized functions (see Sec. "Generalized functions").

♢ The derivation of an equation of a vibrating string calls for a number of mechanical and geometrical assumptions. The question whether the equation describes a physical process adequately enough can be answered only by comparing the results obtained on solving the equation and experimental data. A similar question can be addressed to other differential equations.

7 The equation of longitudinal vibrations for strings and rods

The equations of longitudinal vibrations of springs, strings, and rods are written in the same form. This problem however can be treated more elearly by the example of an elastic rod. We shall call a rod such a solid body whose dimension across is small enough compared to the dimension lengthwise (for a string its dimension across is negligible). When deriving the equation, we shall assume that the tensile stresses appearing during vibrations obey the Hooke law. It is the conditions of the fulfillment of the Hooke law (elastic tension) that, in fact, determine the region of applicability of the sought-for equation.

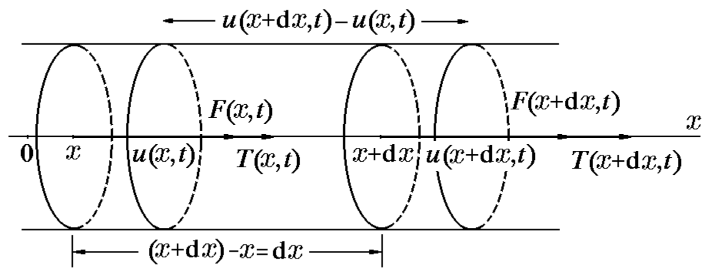

Fig. 29.

Let the coordinate axis Ox coincide with the direction of the longitudinal axis of an elastic rod. By longitudinal vibrations will be meant the displacements of the cross (perpendicular to the Ox-axis) sections of the rod, $S(x)$, along the Ox-axis (Fig. 29), such that the cross sections under considerations remain plane and orthogonal to the Ox-axis in the course of displacement.

Denote by $u(x,t)$ the displacement, at time t, of the rod cross section $S(x)$ that, when being at rest, had the abscissa x. The geometric coordinate x chosen in this way is called a Lagrangian variable. This choice is natural, but not the only possible. Thus, for instance, as a geometric coordinate we can use the quantity $X = x+u$, called an Eulerian coordinate (on the relation between Lagrangian and Eulerian coordinates see [18]).

Let $\rho = \rho(x)$ be the density of the rod in the nonperturbed state; $F = F(x,t)$ be the spatial density of the external forces acting strictly along the Ox-axis; $E = E(x)$ be Young's modulus, and $T = T(x,t)$ the tension.

Calculate the relative elongation of a segment $(x, x + dx)$ at time t. Since the coordinates of the ends of this segment at the time t are $[x+u(x,t)$ and $x+dx+u(x+$

$dx, t)]$, the relative elongation equals (see Fig. 29)

$$\frac{\{[x + dx + u(x + dx, t)] - [x + u(x, t)]\} - \{(x + dx) - x\}}{dx} =$$
$$= \frac{u(x + dx, t) - u(x, t)}{dx} = \frac{\partial u(x, t)}{\partial x}.$$

In view of the fact that the tension $T(x, t)$ is proportional to the relative elongation, we find

$$T(x, t) = E(x)\frac{\partial u(x, t)}{\partial x}. \tag{7.1}$$

Consider the rod segment $dV = S(x)dx$ confined between the cross sections $S(x)$ and $S(x+dx)$. Let the increment dx be such that the functions $S(x)$, $F(x, t)$, and $\rho(x)$ can be considered to be constant within this segment. Then the counterdirected tensile forces $\mathcal{F}_1(x, t) = T(x, t)S(x)$ and $\mathcal{F}_2(x, t) = T(x + dx, t)S(x + dx)$ and the external force $\mathcal{F}_3(x, t) = F(x, t)S(x)dx$ act along the Ox-axis. According to Newton' law, a rod segment dV of mass $dm = \rho(x)S(x)dx$ acquires an acceleration $\partial^2 u(x, t)/\partial t^2$, such that the quantity

$$\rho(x)S(x)\frac{\partial^2 u(x, t)}{\partial t^2}dx$$

is equal to the sum of all forces acting on the segment in the direction of its displacement, that is,

$$\rho(x)S(x)\frac{\partial^2 u(x, t)}{\partial t^2}dx = [T(x + dx, t)S(x + dx, t) - T(x, t)S(x, t)] + S(x, t)F(x, t)dx.$$

Then, in view of (7.1),

$$\frac{(TS)(x + dx, t) - (TS)(x, t)}{dx} =$$
$$= \frac{(ES\frac{\partial u}{\partial x})(x + dx, t) - (ES\frac{\partial u}{\partial x})(x, t)}{dx} = \frac{\partial\left(E(x)S(x)\frac{\partial u(x,t)}{\partial x}\right)}{\partial x}$$

and we get the differential equation for longitudinal vibrations of a rod:

$$\rho(x)S(x)\frac{\partial^2 u(x, t)}{\partial t^2} = \frac{\partial}{\partial x}\Big(S(x)E(x)\frac{\partial u(x, t)}{\partial x}\Big) + S(x)F(x, t). \tag{7.2}$$

In case of a rod or a string with constant cross section, $S(x) = \text{const}$, from Eq. (7.2) we obtain

$$\rho(x)\frac{\partial^2 u(x, t)}{\partial t^2} = \frac{\partial}{\partial x}\Big(E(x)\frac{\partial u(x, t)}{\partial x}\Big) + F(x, t). \tag{7.3}$$

If, in addition, the rod or the string are homogeneous, that is, $\rho(x)$ and $E(x)$ are constants, Eq. (7.3) will be even simpler, taking the form

$$\frac{\partial^2 u(x, t)}{\partial t^2} = a^2\frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t), \tag{7.4}$$

where $a^2 = E/\rho$ and $f(x, t) = F(x, t)/\rho$. This equation completely coincides with the one-dimensional wave equation (6.3). Therefore, the additional conditions for finding a unique solution of Eq. (7.4) completely coincide with those for Eq. (6.3).

Equations (6.3) and (7.4), being second order partial differential equations, can be written as a system of two first order differential equations. Actually, putting in (7.4)

$$W(x,t) = \frac{\partial u(x,t)}{\partial t} \quad \text{and} \quad T(x,t) = E(x)\frac{\partial u(x,t)}{\partial x},$$

we obtain

$$\begin{cases} \dfrac{\partial W}{\partial x} = \dfrac{1}{E}\dfrac{\partial T}{\partial t}, \\ \dfrac{\partial T}{\partial x} = \rho\dfrac{\partial W}{\partial t}. \end{cases} \tag{7.5}$$

As already mentioned, the region of applicability of Eq. (7.4) coincides with that of Hooke's law. In a more general situation

$$T = E\Big(x, \frac{\partial u}{\partial x}\Big)\frac{\partial u}{\partial x},$$

and this leads to a second order quasilinear equation

$$\frac{\partial}{\partial x}\Big[E\Big(x, \frac{\partial u}{\partial x}\Big)\frac{\partial u}{\partial x}\Big] = \rho\frac{\partial^2 u}{\partial t^2} - F(x,t).$$

8 Equations of electric oscillations in wires (telegraph equations)

Let us arrange a wire, which carries an alternating current $i(x,t)$ at an applied voltage $v(x,t)$, along the Ox-axis. Denote by R, L, C, and G, respectively, distributed (per unit length) ohmic resistance, inductance, capacitance, and charge losses through imperfect insulation, assuming the losses to be proportional to the voltage at a given point. To be more specific, suppose that the current direction coincides with that of the Ox-axis.

In accordance with Ohm's law, for a segment of a circuit with coordinates $(x, x + dx)$, we can write that the voltage drop across the circuit segment of length dx is equal to the sum of the electromotive forces:

$$v(x,t) - v(x+dx,t) = Ri(x,t)dx + L\frac{\partial i(x,t)}{\partial t}dx.$$

In view of the relation

$$\frac{v(x+dx,t) - v(x,t)}{dx} = \frac{\partial v(x,t)}{\partial x},$$

this equality takes the form

$$\frac{\partial v(x,t)}{\partial x} = Ri(x,t) + L\frac{\partial i(x,t)}{\partial t}. \tag{8.1}$$

Equating the charge

$$[i(x,t) - i(x+dx,t)]dt = -\frac{\partial i(x,t)}{\partial x}dx\,dt$$

that is passed by the circuit segment $(x, x+dx)$ during the time from t to $t+dt$ to the charge

$$C[v(x,t+dt)-v(x,t)]dx + Gv(x,t)dx\,dt = \Big[C\frac{\partial v(x,t)}{\partial t} + Gv(x,t)\Big]dx\,dt$$

that is expended for charging the circuit element $(x, x+dx)$ and the leakage current due to imperfect insulation, we find

$$\frac{\partial i(x,t)}{\partial x} + C\frac{\partial v(x,t)}{\partial t} + Gv(x,t) = 0. \tag{8.2}$$

Relations (8.1) and (8.2) are called *the telegraph equations* and form a system of first order partial differential equations. This system of equations can be reduced to one second order partial differential equation for the current $i(x,t)$:

$$\frac{\partial^2 i(x,t)}{\partial x^2} = CL\frac{\partial^2 i(x,t)}{\partial t^2} + (CR+GL)\frac{\partial i}{\partial t} + GRi(x,t) \tag{8.3}$$

or for the voltage:

$$\frac{\partial^2 v(x,t)}{\partial x^2} = CL\frac{\partial^2 v(x,t)}{\partial t^2} + (CR+GL)\frac{\partial v}{\partial t} + GRv(x,t). \tag{8.4}$$

Equations (8.3) and (8.4) are absolutely identical in form and are called the telegraph equations as well. Within the framework of electrodynamics, the equations obtained can be considered as rather good approximations that do not take into account the electromagnetic oscillations in the medium surrounding the wire.

If in Eqs. (8.1)–(8.4) we neglect the losses through the insulation and assume that the ohmic resistance $G = R \approx 0$, then (8.1) and (8.2) take the form

$$\begin{cases} -\dfrac{\partial i}{\partial x} = C\dfrac{\partial v}{\partial t}, \\ \dfrac{\partial v}{\partial x} = L\dfrac{\partial i}{\partial t}, \end{cases} \tag{8.5}$$

and Eqs. (8.3) and (8.4) are reduced to the well-known equations for an oscillatory circuit

$$\frac{\partial^2 i}{\partial t^2} = a^2\frac{\partial^2 i}{\partial x^2}, \qquad \frac{\partial^2 v}{\partial t^2} = a^2\frac{\partial^2 v}{\partial x^2}, \qquad a = \frac{1}{\sqrt{LC}}, \tag{8.6}$$

which are completely identical in form to the one-dimensional wave equation.

The general scheme of the formulation of the initial and boundary value problems for the telegraph equations completely coincides with that given in Sec. "The equation for vibrating strings" (the Cauchy problem for an infinitely long wire, mixed problems for finite and semi-infinite wires, etc.). One should also remember that when formulating boundary conditions for the segments $]a, a+dx[$ and $]b-dx, b[$, it is necessary to consider the voltage drop and the discharge intake instead of the equations of motion. If a circuit contains a lumped resistor of ohmic resistance R, an inductor of inductance $L_{\rm c}$, and a capacitor of capacitance $C_{\rm c}$, connected in series, then the voltage drop is given by the formula

$$\Delta v = R_{\rm c}i + L_{\rm c}\frac{\partial i}{\partial t} + \frac{1}{C_{\rm c}}\int i dt.$$

It is of interest to compare the system of equations that describe longitudinal vibrations of a rod, Eqs. (7.5), with the system of telegraph equations (8.5)

$$\begin{cases} -\dfrac{\partial v}{\partial x} = \dfrac{1}{E}\dfrac{\partial T}{\partial t}, \\ \dfrac{\partial T}{\partial x} = \rho\dfrac{\partial v}{\partial t}; \end{cases} \qquad \begin{cases} -\dfrac{\partial i}{\partial x} = C\dfrac{\partial v}{\partial t}, \\ -\dfrac{\partial v}{\partial x} = L\dfrac{\partial i}{\partial t}. \end{cases} \tag{8.7}$$

There is an obvious physical analogy between the voltage v and the tension T of the rod, between the electric current i and the mechanical velocity w. A similar relation can be established between the mechanical and the electrical characteristics of physical processes. Thus, the inductance of an electric circuit, L, is an analog to the density of a solid and the capacitance C is an analog to the inverse modulus of elasticity $1/E$. In view of the fact that both systems of equations (8.7) were obtained under the assumptions that disregard the ohmic resistance and the resistance of the medium, the mechanical resistance is an analog to the ohmic resistance. This coincidence of mathematical descriptions of various physical problems makes it possible to model and study mechanical systems with the help of electrical systems and *vice versa*.

9 The equation of transverse vibrations of membranes

By a membrane we shall mean a solid body whose thickness is negligible compared to its other dimensions, which does not resist to bending, or, in other words, a thin ideally flexible film.

Suppose that a membrane is stretched uniformly in all directions and, being in equilibrium, occupies some region S with boundary L in the xOy-plane. Denote by $u(x, y, t)$ the displacement of the membrane point with coordinates (x, y) at time t in the direction orthogonal to the xOy-plane under the action of external forces with density $F(x, y, t)$, which are directed perpendicular to the xOy-plane. Then the expression $u = u(x, y, t)$ for a fixed time t can be considered as an equation of the bent surface taken by the membrane in the process of vibration.

Let $d\sigma$ be an element of this surface with a unit normal vector

$$\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),$$

and dS be the projection of $d\sigma$ on the xOy-plane. Then

$$d\sigma = \frac{dS}{\cos\theta} = \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}\, dx\, dy. \tag{9.1}$$

A segment dl of the boundary of $d\sigma$ is subject to tension $\vec{T}(x, y, t)dl$. Since the membrane does not resist to bending, the tension vector $\vec{T}(x, y, t)$ is located in a plane orthogonal to the normal vector $\vec{n}$ (a plane tangential to $u = u(x, y, t)$). If the vector $\vec{T}(x, y, t)$ forms with the xOy-plane an angle θ', then this angle does not exceed the angle θ between the normal $\vec{n}$ and the Ou-axis, that is,

$$\cos\theta' \geqslant \cos\theta = 1\Big/\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2}. \tag{9.2}$$

Neglect in relations (9.1) and (9.2) infinitesimals of higher order with respect to $\partial u/\partial x$ and $\partial u/\partial y$, then $d\sigma \approx dS$. Thus, the membrane area remains almost unchanged in the process of vibrations and this, in turn, means that the tension vector $\vec{T}(x, y, t)$, according to Hooke's law, may change only its direction, remaining constant in absolute value: $|\vec{T}(x, y, t)| = T_0 = \text{const}$.

In view of this, the projections T_{xOy} and T_{Ou} of the tension vector on the xOy-plane and on the Ou-axis can be written in the form

$$T_{xOy} = T_0 \cos\theta' \approx T_0, \qquad T_{Ou} = T_0 \frac{\partial u}{\partial n}.$$

Denote the surface density of the membrane by $\rho(x, y)$ and the projection of an arbitrary section of the bent surface of the membrane on the xOy-plane with boundary L by S_1 and equate the momenta of the vertical tensile forces and the momenta of the external forces in a time $t_2 - t_1$ to the change in momentum of the selected section of the membrane:

$$\int_{t_1}^{t_2} dt \Big\{ \int_{L_1} T_0 \frac{\partial u}{\partial n} dl + \iint_{S_1} F dxdy \Big\} = \iint_{S_1} \Big[\frac{\partial u(x, y, t_2)}{\partial t} - \frac{\partial u(x, y, t_1)}{\partial t} \Big] \rho(x, y) dxdy.$$

Assuming that $t_2 - t_1$ is small, $t_2 - t_1 \approx dt$, and following the Lagrange mean-value theorem, we can write

$$\frac{\partial u(x, y, t_2)}{\partial t} - \frac{\partial u(x, y, t_1)}{\partial t} = \frac{\partial^2 u(x, y, t)}{\partial t^2} dt.$$

Using Green's formula

$$\int_{L_1} \frac{\partial u}{\partial n} dl = \iint_{S_1} \Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \Big) dxdy,$$

we obtain an integral equation of vibrations

$$\int_{t_1}^{t_2} dt \iint_{S_1} \Big[\rho \frac{\partial^2 u}{\partial t^2} - T_0 \Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \Big) - F(x, y, t) \Big] dxdy = 0.$$

From here, in view of the arbitrariness of S_1 and $t_2 - t_1$, we arrive at the differential equation of transverse vibrations of membrane

$$\rho \frac{\partial^2 u}{\partial t^2} = T_0 \Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \Big) + F(x, y, t). \tag{9.3}$$

In case of a homogeneous membrane, $\rho(x, y) = \rho = \text{const}$ and we get an equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \Big) + f(x, y, t), \tag{9.4}$$

where $a^2 = T_0/\rho$ and $f(x, y, t) = F(x, y, t)/\rho$, which is called the two-dimensional wave equation.

The statement of the problems of finding unique solutions to the two-dimensional wave equation and the classification of these problems completely coincide with those considered for the one-dimensional wave equation. The differences arising when formulating the boundary conditions for a plane curve L do not lead to crucial difficulties.

Note that the choice of specific models (string, membrane, *etc.*) and their consideration are conditioned by striving for the most simple and ocular demonstration of the principal scheme for deriving equations of mathematical physics, which illustrates how mathematical and physical aspects of the problems go well together and complement one another.

Actually, all equations derived above can be obtained from the Lamé equation

$$\rho\frac{\partial^2\vec{u}}{\partial t^2} = (\lambda + 2\mu)\,\mathrm{grad}\,\mathrm{div}\,\vec{u} - \mu\,\mathrm{rot}(\mathrm{rot}\,\vec{u}) + \vec{F} \tag{9.5}$$

for the three-dimensional displacement vector $\vec{u} = (u(x,y,z,t), v(x,y,z,t), w(x,y,z,t))$, which is a special subject of investigation in elasticity theory. However, the derivation of this equation, notwithstanding the cumbersome calculations, is based on Hooke's law and the above scheme. First, in a homogeneous isotropic solid body of density ρ, a volume element with the facet coordinates $x, x+dx$, $y, y+dy$, and $z, z+dz$ is singled out which, under the action of internal forces (stresses) and external forces of density $\vec{F} = (X(x,y,z,t), Y(x,y,z,t), Z(x,y,z,t))$, is displaced by $\vec{u}(x,y,z,t)$. Then three equations for the projections on each coordinate axis are written for the singled out element, such that the changes in momentum are equated with the impulse of internal and external forces for a time dt. As a result, we have for displacements the system of equations

$$\begin{aligned}
\rho\frac{\partial^2 u}{\partial t^2} &= G\Big(\Delta u + \frac{m}{m-2}\frac{\partial}{\partial x}(\mathrm{div}\,\vec{u})\Big) + X,\\
\rho\frac{\partial^2 v}{\partial t^2} &= G\Big(\Delta v + \frac{m}{m-2}\frac{\partial}{\partial y}(\mathrm{div}\,\vec{u})\Big) + Y,\\
\rho\frac{\partial^2 w}{\partial t^2} &= G\Big(\Delta w + \frac{m}{m-2}\frac{\partial}{\partial z}(\mathrm{div}\,\vec{u})\Big) + Z,
\end{aligned} \tag{9.6}$$

where G is the shear modulus and m is the Poisson coefficient that characterizes the ratio of the corresponding transverse compression to the longitudinal tension (see [9]). With the help of the Lamé coefficients $\mu = G$ and $\lambda = 2G/(m-2)$, Eqs. (9.6) can be written as a vector equation [Eq. (9.5)].

If we seek a solution to Eq. (9.5) in the form

$$\vec{u} = \mathrm{grad}\,A_0 + \mathrm{rot}\,\vec{A},$$

then the scalar potential A_0 and the vector potential $\vec{A}$ satisfy the three-dimensional wave equations

$$\begin{aligned}
\frac{\partial^2 A_0}{\partial t^2} &= a^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\Big)A_0 + \frac{1}{\rho}\Phi_0,\\
\frac{\partial^2 \vec{A}}{\partial t^2} &= b^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\Big)\vec{A} + \frac{1}{\rho}\vec{\Phi},
\end{aligned} \tag{9.7}$$

where Φ_0 and $\vec{\Phi}$ are the scalar and the vector potential of the vector field $\vec{F}$

$$\vec{F} = \mathrm{grad}\,\Phi_0 + \mathrm{rot}\,\vec{\Phi}$$

and

$$a^2 = \frac{\lambda + 2\mu}{\rho}, \qquad b^2 = \frac{\mu}{\rho}.$$

Along with the foregoing, variational methods are successfully used in classical mechanics (see Sec. "Generalized functions"). Let us consider the variational method of deriving, for example, the equilibrium equation for an elastic membrane under the action of stationary external forces $F(x, y)$. Let $A[u]$ be the functional that defines the work of the external and elastic forces on the displacement of the membrane from the equilibrium position, $u = 0$, to the position of a bent surface, $u = u(x, y)$. Since the work of an external force is given by

$$\iint_S F(x, y)u(x, y)dxdy,$$

and the work of elastic forces, according to (9.1), is

$$-T_0 \iint_S \Big[\sqrt{1 + \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial y}\Big)^2} - 1\Big]dx\,dy \approx -\frac{T_0}{2} \iint_S \Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial y}\Big)^2\Big]dx\,dy,$$

then $A[u]$ has the form

$$A[u] = \iint_S \Big\{Fu - \frac{T_0}{2}\Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial y}\Big)^2\Big]\Big\}dx\,dy, \tag{9.8}$$

and, hence, the variation of the functional (9.8) is determined by the expression

$$\delta A[u] = \iint_S \Big\{F\delta u - T_0\Big[\frac{\partial u}{\partial x}\delta\Big(\frac{\partial u}{\partial x}\Big) + \frac{\partial u}{\partial y}\delta\Big(\frac{\partial u}{\partial y}\Big)\Big]\Big\}dx\,dy. \tag{9.9}$$

For the membrane being in equilibrium, $\delta A(u) = 0$. Then, in view of the fact that

$$\delta\Big(\frac{\partial u}{\partial x}\Big) = \frac{\partial}{\partial x}(\delta u), \qquad \delta\Big(\frac{\partial u}{\partial y}\Big) = \frac{\partial}{\partial y}(\delta u)$$

and that the variation vanishes on the boundary, $\delta u|_L = 0$, we obtain

$$\iint_S \Big\{F + T_0\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big)\Big\}\delta u\,dxdy - T_0 \int_L \frac{\partial u}{\partial n}\delta u\,dl =$$
$$= \iint_S \Big\{F + T_0\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big)\Big\}\delta u\,dxdy = 0,$$

where $\vec{n}$ is an external normal to the curve L. In view of the arbitrariness of δu and in accordance with the principal lemma of calculus of variations, we obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{F(x, y)}{T_0} = 0.$$

Naturally, a similar equation follows from (9.4) and (9.5).

10 Equations of hydrodynamics and acoustics

Consider the motion of a fluid which occupies a certain volume. Select some portion of the fluid, a "droplet", confined in a volume ΔV. If the volume ΔV is small compared to the dimensions of the system and the forces that act on the fluid portions confined in this volume can be considered constant, we arrive at the notion of a material point of fluid. If we may neglect the friction forces between fluid particles or, in other words, viscosity, we arrive at the notion of ideal fluid.

Let, in a Cartesian coordinate system,

$$\vec{v} = \frac{d\vec{x}}{dt}, \qquad \vec{x} = (x_1, x_2, x_3), \tag{10.1}$$

be the velocity vector for a moving fluid that defines the trajectory of motion of every material point of the fluid. Let $\rho(\vec{x}, t)$ and $p(\vec{x}, t)$ be the fluid density and pressure at point $\vec{x}$ at time t and $G(\vec{x}, t)$ and $F(\vec{x}, t)$ be the intensities of the sources of mass forces.

Let us single out some closed surface S with an external normal vector $\vec{n}$, which encloses a volume V. The variation in quantity of fluid per unit time is equal to the fluid flux through the boundary S and the inflow of matter from internal sources

$$\frac{d}{dt}\iiint\limits_V \rho(\vec{x}, t)dV = -\iint\limits_S (\vec{v}(\vec{x}, t), d\vec{S})\rho(\vec{x}, t) + \iiint\limits_V G(\vec{x}, t)dV,$$

where $d\vec{S} = \vec{n}dS$. Transforming the surface integral to a volume one by the Ostrogradskii–Gauss formula, we obtain

$$\iiint\limits_V \Big[\frac{\partial\rho(\vec{x}, t)}{\partial t} + \operatorname{div}(\rho(\vec{x}, t)\vec{v}) - G(\vec{x}, t)\Big]dV = 0.$$

In view of the arbitrariness of the volume V, we arrive at the differential equation

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\vec{v}) = G(\vec{x}, t). \tag{10.2}$$

♦ Equation (10.2) is called the continuity equation or the transport equation.

Another equation which characterizes the motion of fluid can be obtained by considering all forces (external forces, pressure forces) acting on an element dV of the volume V. Thus, the resulting pressure force is

$$\vec{F}_1(t) = \iint\limits_S p(\vec{x}, t)\vec{n}\,dS = \iiint\limits_V \operatorname{grad} p(\vec{x}, t)\,dV,$$

and the resultant of all mass forces $\vec{F}(\vec{x}, t)$

$$\vec{F}_2(t) = \iiint\limits_V \rho(\vec{x}, t)\vec{F}(\vec{x}, t)\,dV.$$

Since from (10.1) it follows that

$$\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + \frac{\partial\vec{v}}{\partial x}v_x + \frac{\partial\vec{v}}{\partial y}v_y + \frac{\partial\vec{v}}{\partial z}v_z = \frac{\partial\vec{v}}{\partial t} + (\vec{v}, \operatorname{grad})\vec{v}, \tag{10.3}$$

the variation in the momentum of the matter in the selected volume is written in the form

$$\iiint\limits_V \rho(\vec{x},t)\frac{d\vec{v}}{dt}dV = \iiint\limits_V \rho(\vec{x},t)\Big[\frac{\partial\vec{v}}{\partial t} + (\vec{v},\mathrm{grad})\vec{v}\Big]dV.$$

Hence, in accordance with Newton's second law, we obtain

$$\iiint\limits_V \rho(\vec{x},t)\Big\{\Big[\frac{\partial\vec{v}}{\partial t} + (\vec{v},\mathrm{grad})\vec{v}\Big] + \frac{1}{\rho(\vec{x},t)}\,\mathrm{grad}\,p(\vec{x},t) - \vec{F}(\vec{x},t)\Big\}dV = 0. \qquad (10.4)$$

In virtue of the arbitrariness of V, we obtain from (10.4) the differential equation of motion of fluid

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v},\mathrm{grad})\vec{v} + \frac{1}{\rho}\,\mathrm{grad}\,p(\vec{x},t) = \vec{F}(\vec{x},t). \qquad (10.5)$$

♦ Equation (10.5) is called *the equation of motion of ideal fluid*, or *Euler's equation.*

The system of four equations consisting of the continuity equation (10.2) and Euler's equation (10.5), contains five unknown functions ($\vec{v}$, p, ρ) and is not closed. Therefore, for a complete and unambiguous description of the process of motion of ideal fluid, the system of equations (10.2), (10.5) must be complemented with an equation of state that relates pressure $p(\vec{x},t)$ and density $\rho(\vec{x},t)$. In the general case, the equation of state contains absolute temperature T (e.g., for a perfect gas

$$\rho = \frac{p}{RT},$$

where R is the gas constant). In this case, a closed system of equations should also include the heat equation (see below) that describes the variation of the system temperature.

In some cases, the equation of state does not involve a temperature dependence, that is,

$$\rho = f(p). \qquad (10.6)$$

For instance, for an incompressible fluid we have

$$\rho = \mathrm{const}\,. \qquad (10.7)$$

The temperature dependence of ρ can be neglected for adiabatic processes, i.e., processes which occur so rapidly that heat has no time to be transferred from one fluid portion to another, and we have

$$\rho = \rho_0\Big(\frac{p}{p_0}\Big)^{1/\gamma}. \qquad (10.8)$$

Here $\gamma = c_p/c_V$ is the ratio of the specific heat at constant pressure to that at constant volume.

Here we shall dwell on the adiabatic motion of gases. In this case, the system of hydrodynamic equations has the form

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = G, \\ \dfrac{\partial \vec{v}}{\partial t} + (\vec{v}, \operatorname{grad})\vec{v} + \dfrac{1}{\rho} \operatorname{grad} p = \vec{F}, \\ \rho = f(p) = \rho_0 \Big(\dfrac{p}{p_0}\Big)^{1/\gamma}. \end{cases} \tag{10.9}$$

This is a system of nonlinear equations which can be "linearized" if some simplifying assumptions are made. Assume that $G(\vec{x}, t) \equiv 0$ and $\vec{F}(\vec{x}, t) \equiv 0$ and that the motion of gas can be simulated as small vibrations about the equilibrium position characterized by constant values of density ρ_0 and pressure p_0. Rejecting the squares, products, and higher degrees of the quantities $\vec{v}$, $\rho - \rho_0$, and $p - p_0$ and their derivatives, we obtain the "linearized" adiabatic equation

$$\rho = \rho_0 + \frac{\rho_0}{\gamma p_0}(p - p_0) + \cdots \approx \rho_0 + \frac{1}{a^2}(p - p_0), \tag{10.10}$$

$$\gamma \frac{p_0}{\rho_0} = a^2;$$

Euler's equation

$$\frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho_0} \operatorname{grad} p, \tag{10.11}$$

and the continuity equation

$$\frac{\partial \rho}{\partial t} + \rho_0 \operatorname{div} \vec{v} = 0, \tag{10.12}$$

which, in view of (10.10), can be represented as

$$\frac{1}{a^2} \frac{\partial p}{\partial t} + \rho_0 \operatorname{div} \vec{v} = 0. \tag{10.13}$$

We are interested in a potential (irrotational) flow of an ideal fluid in which every small volume is deformed and moves translationally without rotation. In this case,

$$\vec{v} = -\operatorname{grad} \varphi. \tag{10.14}$$

The function $\varphi(\vec{x}, t)$ is called a velocity potential. Let us show that, despite the fact that relation (10.14) defines the potential up to an arbitrary function of time, it suffices to know this potential to describe completely the whole motion process. Actually, substitution of (10.14) into (10.11) yields

$$-\operatorname{grad} \frac{\partial \varphi}{\partial t} = -\frac{1}{\rho_0} \operatorname{grad} p$$

or

$$p = \rho_0 \frac{\partial \varphi}{\partial t}. \tag{10.15}$$

Substituting (10.15) and (10.14) into (10.13), we obtain the equation

$$\frac{\partial^2 \varphi}{\partial t^2} - a^2 \Delta \varphi = 0 \tag{10.16}$$

that should be satisfied by the potential φ. This is a three-dimensional wave equation, and its solution obtained with the help of relations (10.15), (10.14), and (10.10) completely defines the functions $\vec{v}$, ρ, and p. On the other hand, differentiating Eq. (10.16) with respect to t, x_1, x_2, and x_3, we see that these quantities themselves satisfy the equations which coincide with (10.16), that is,

$$\begin{aligned} &\frac{\partial^2 \vec{v}}{\partial t^2} - a^2 \Delta \vec{v} = 0, \\ &\frac{\partial^2 p}{\partial t^2} - a^2 \Delta p = 0, \\ &\frac{\partial^2 \rho}{\partial t^2} - a^2 \Delta \rho = 0. \end{aligned} \tag{10.17}$$

Equations (10.16) and (10.17) are called the acoustic equations or the hydrodynamic equations in the acoustic approximation, because, under the assumptions made, they offer a rather good description of processes with small variations of density and pressure, for example, the propagation of sound. (As shown below, in physics the coefficient $a = \sqrt{\gamma p_0/\rho_0}$ corresponds to the velocity of propagation of vibrations, which, for normal atmospheric conditions, yields the sound velocity $a = 335$ m/s.) The boundary conditions for the wave equations (10.17) are formulated by a routine scheme. For instance, for an impenetrable boundary S with an external normal $\vec{n}$

$$(\vec{v}, \vec{n})\Big|_S = 0 \quad \text{or} \quad \frac{\partial \varphi}{\partial n}\Big|_S = 0.$$

For processes which are characterized by high velocities and high pressure gradients (supersonic flow-around, explosive waves, etc.) the "acoustic approximation" is inadequate and it is necessary to use nonlinear equations of hydrodynamics (gas dynamic), which are beyond the scope of this course.

11 Equation of heat conduction in rod

Let the longitudinal axis of a rod be directed along the Ox-axis and $u(x, t)$ be the temperature of all points of the rod cross section passing through the point with coordinate x. Consider a rod segment confined between the cross sections with coordinates x and $x + dx$ (Fig. 30).

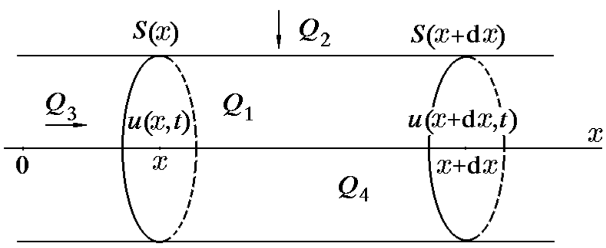

Fig. 30.

Denote by Q_1 the amount of heat expended to heat the selected segment and by Q_2 the amount of heat incoming through the lateral surface of the segment. According to Newton's law, Q_2 is proportional to the temperature difference at the lateral surface, and the sign of Q_2 is conditioned by the sign of this difference. Also denote by Q_3 the amount of heat incoming through the cross section of the rod and determined by Fourier's law and by Q_4 the amount of heat incoming from internal heat sources (e.g., chemical reactions). Thus, the heat balance equation for this rod element and for a time interval dt has the form

$$Q_1 = Q_2 + Q_3 + Q_4. \tag{11.1}$$

Let $S(x)$ and $p(x)$ be, respectively, the area and the perimeter of the rod cross section; $\rho(x)$, $c(x)$, and $k(x)$ be, respectively, the density, the heat capacity, and the heat conductivity of the rod material; $F(x,t)$ be the volume power density of the internal heat sources, and u_0 and k_0 be, respectively, the temperature and the heat conductivity of the external medium.

In view of the above notations, the quantities Q_i, $i=\overline{1,4}$, can be written as

$$\begin{aligned}
&Q_1 = c(x)\rho(x)S(x)[u(x,t+dt)-u(x,t)]dx = c(x)\rho(x)S(x)\frac{\partial u(x,t)}{\partial t}dxdt,\\
&Q_2 = p(x)k_0(x)[u_0(x,t)-u(x,t)]dxdt,\\
&Q_3 = [(kSu_x)(x+dx,t)-(kSu_x)(x,t)]dt = \frac{\partial}{\partial x}\Big(k(x)S(x)\frac{\partial u(x,t)}{\partial x}\Big)dx\,dt,\\
&Q_4 = F(x,t)S(x)dxdt.
\end{aligned} \tag{11.2}$$

Substituting (11.2) into (11.1), we obtain the differential equation of heat conduction in a rod

$$c\rho S\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(kS\frac{\partial u}{\partial x}\Big) + pk_0(u_0-u) + FS. \tag{11.3}$$

If the cross section of the rod is invariable, this equation reduces to

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(k\frac{\partial u}{\partial x}\Big) + F + \frac{pk_0}{S}(u_0-u). \tag{11.4}$$

If the rod is homogeneous, i.e., ρ, c, and k are constant, Eq. (11.4) can be written as

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} - bu + f. \tag{11.5}$$

Here $a^2 = k/(c\rho)$ is a constant called a thermal conductivity coefficient; $h = k_0p/S$ is the coefficient of the heat exchange with the surrounding medium through the lateral surface;

$$b = \frac{h}{c\rho}, \qquad f = \frac{F}{c\rho} + bu_0.$$

If there is no heat exchange with the surrounding medium through the lateral surface (the lateral surface is heat-insulated), i.e., $h=0$, Eq. (11.5) reduces to

$$\frac{\partial u}{\partial t} = a^2\frac{\partial u^2}{\partial x^2} + f, \tag{11.6}$$

which is called the one-dimensional heat equation.

To find a unique solution of Eq. (11.6), it is necessary to preset the initial temperature distribution and the heat regime at the ends of the rod.

Let us consider classical statements of initial and boundary value problems for Eq. (11.6).

1. *The Cauchy problem*

In contrast to the one-dimensional wave equation, the heat equation calls for only one initial condition

$$u(x,t)|_{t=0} = \varphi(t). \tag{11.7}$$

The problem of finding the temperature described by Eq. (11.6) in combination with the initial condition (11.7) is a Cauchy problem. Problems of this type describe rather adequately actual physical processes for which it is required to find the temperature distribution in a rod either for a short time interval (such that the boundary conditions have no time to influence substantially the process) or for a very long (ideally, infinitely long) rod where the boundary conditions have no effect on the temperature in the central part of the rod under consideration. However, it should be noted that the boundary conditions left unsaid for these problems have the form

$$\lim_{x\to\pm\infty} u(x,t) = 0,$$

which follows from the energy conservation law.

2. *The mixed problem*

Let us consider a rod whose ends are located at points with coordinated $x = a$ and $x = b$. In this case, the initial condition should be complemented with boundary conditions. Let us consider three principal (classical) kinds of boundary conditions.

(a) **Boundary conditions of the first kind.** The sought-for function $u(x,t)$ is specified at the end point of the rod $x = a$, i.e.

$$u(x,t)\Big|_{x=a} = \mu(t),$$

which may be a constant. In particular, if the temperature at the rod end is kept equal to zero, then $\mu(t) = 0$.

(b) **Boundary conditions of the second kind.** The partial derivative of the function $u(x,t)$ is specified at one end point of the rod, i.e.,

$$\frac{\partial u(x,t)}{\partial x}\Big|_{x=a} = \nu(t).$$

Let us consider the physical sense of this condition. If we specify at the rod boundary, e.g., at the point $x = a$, the heat flux instead of the temperature, then, according to Fourier's law,

$$Q(t) = k\frac{\partial u(x,t)}{\partial x}\Big|_{x=a}.$$

Denoting $\nu(t) = Q(t)/k$, we arrive at the condition for the derivative. When $\nu(t) \equiv 0$ the rod end is heat-insulated.

(c) **Boundary conditions of the third kind.** On the rod boundary, a linear combination of the function $u(x,t)$ and its derivative is specified:

$$\left(\frac{\partial u(x,t)}{\partial x} + \alpha u(x,t)\right)\Big|_{x=a} = \gamma(t). \tag{11.8}$$

This condition corresponds to the heat exchange (by Newton's law) with the surrounding medium of given temperature $T_0(t)$ at the rod end $x = a$. Actually, the heat flux at the boundary can be written both as

$$Q = k\frac{\partial u(x,t)}{\partial x}\Big|_{x=a},$$

and as $Q = h[T_0(t) - u(x,t)]\big|_{x=a}$. Then

$$k\frac{\partial u(x,t)}{\partial x}\bigg|_{x=a} = h[T_0(t) - u(x,t)]\big|_{x=a}.$$

It should be noted that, in contrast to the preceding case, the heat flux is "self-controlled" depending on the difference $[T_0(t) - u(x,t)]\big|_{x=a}$: it decreases as the temperatures on the external and the internal sides of the boundary become equal. Introducing the notations $\alpha = h/k$ and $\gamma(t) = hT_0/k$, we arrive at the condition (11.8).

The boundary conditions at the point $x = b$ are considered analogously. The boundary conditions at the points $x = a$ and $x = b$ may be of different kind. When $x = b \to \infty$ we come to the problem for a semi-infinite rod.

The problem of finding a solution of Eq. (11.6) with the initial condition (11.7) and the boundary conditions (a), (b), or (c) is called the *mixed* problem for the heat equation. Boundary conditions other than the classical ones (a), b), and (c) can be obtained by describing the heat balance for the boundary segments of the rod $]a, a+dx[$ and $]b - dx, b[$, as was done in deriving the heat equation. Let, for example, the rod end $x = a$ be clamped in a massive lock having a high heat conductivity and heat capacity c_0. Then the boundary condition will be written as

$$\Big[u_x(x,t) - \frac{c_0}{k}u_t(x,t)\Big]\Big|_{x=a} = 0.$$

Along with the problem statement with linear boundary conditions, it is possible to formulate problems with nonlinear boundary conditions. Thus, for instance, account for the emission of radiation by the Stefan–Boltzmann law from the end $x = a$ into a medium with the temperature $u_0(t)$ leads to the following boundary condition:

$$k\frac{\partial u(x,t)}{\partial x}\bigg|_{x=a} = H[u^4(t) - u_0^4(x,t)]\big|_{x=a}.$$

3. *The boundary value problem*

The influence of the initial conditions in a rod weakens with time due to the redistribution of heat. If the observation time t can be considered long enough (much longer than the relaxation time), the temperature of the rod is determined, within the measurement error, in fact only by the boundary conditions. Therefore, the contribution of the initial conditions (11.7) becomes inessential, and we arrive at a purely boundary value problem of finding a solution of Eq. (11.6) for $a \leqslant x \leqslant b$, $t \to \infty$. This is the first (second, third) boundary value problem if the boundary conditions are of the first (second, third) kind.

In this statement, we may set off two interesting problems: the Fourier problem of temperature waves and the problem of a stationary distribution of heat. In the first case, the first boundary value problem for a semi-infinite rod with a periodic boundary condition is considered. In the second case, any boundary value problem is considered on the assumption that the density of the internal heat sources and the boundary conditions are independent of time. In this case, the one-dimensional heat equation reduces to the ordinary differential equation

$$a^2\frac{d^2u(x)}{dx^2} + f(x) = 0$$

or, in a more general statement [see Eq. (11.3)], to the equation

$$\frac{d}{dx}\Big(kS\frac{du}{dx}\Big) + pk_0(u_0 - u) + FS = 0.$$

In conclusion, it should be noted that taking into account the temperature dependence of the physical characteristics ρ, c, and k results in a quasilinear equation

$$c(x,u)\rho(x,u)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(k(x,u)\frac{\partial u}{\partial x}\Big) + F(x,t), \qquad (11.9)$$

which substantially extends the region of applicability of the heat equation.

12 Equation of diffusion and heat conduction in space

The problem of heat conduction is akin to the problem of diffusion, i.e., variations in concentration of a substance. If a substance is originally nonuniformly distributed in a solution or a gas, there occurs its redistribution from regions of higher concentration to regions of lower concentration. This process is called diffusion and obeys Nernst's law, which is similar to Fourier's law for heat conduction.

Let $u(\vec{x},t)$ be the concentration of a substance at a point $\vec{x} = (x,y,z)$ at time t that is distributed in an immobile isotropic medium that takes some volume with the porosity coefficient $\gamma(\vec{x})$ (ratio of the volume of pores to the total volume) and the diffusion coefficient $D(\vec{x})$. (For an anisotropic medium, the latter are tensors rather than scalars.) For diffusion occurring with absorption or release of the given substance, for instance, due to chemical reactions, let $F(\vec{x},t)$ be the spatial power density of the sources and sinks of the substance.

Select inside the volume under consideration an arbitrary surface S with a unit external normal $\vec{n}$, which bounds some volume V.

Compose the balance equation for the diffusing substance. Denote by dQ_1 the amount of substance passing through a surface element dS in a unit time. According to Nernst's law,

$$dQ_1 = -D(\vec{x})\frac{\partial u}{\partial n}dS = -(\vec{n},\nabla u)D(\vec{x})dS.$$

Then the amount of substance coming into the volume V through the surface S during a time interval from t_1 to t_2, in view of the Ostrogradskii-Gauss formula, will be given by

$$Q_1 = \int\limits_{t_1}^{t_2} dt \oint\limits_S (\vec{n},\nabla u)D(\vec{x})dS = \int\limits_{t_1}^{t_2} dt \int\limits_V \operatorname{div}(D(\vec{x})\nabla u)d\vec{x}.$$

The amount of substance incoming from internal sources is defined as

$$Q_2 = \int\limits_{t_1}^{t_2} dt \int\limits_V F(\vec{x},t)d\vec{x}.$$

The total inflow of substance, Q_3, resulting from the change in concentration $u(\vec{x},t)$ in a short time Δt, by the Lagrange theorem,

$$u(\vec{x},t+\Delta t) - u(\vec{x},t) \approx \frac{\partial u(\vec{x},t)}{\partial t}\Delta t$$

will be

$$Q_3 = \int_{t_1}^{t_2} dt \int_V \gamma(\vec{x}) \frac{\partial u}{\partial t} d\vec{x}.$$

Since

$$Q_1 + Q_2 - Q_3 = 0,$$

we have

$$\int_{t_1}^{t_2} dt \int_V \Big[\operatorname{div}(D(\vec{x})\nabla u) + F(\vec{x}, t) - \gamma(\vec{x}) \frac{\partial u}{\partial t} \Big] d\vec{x} = 0.$$

By virtue of the arbitrariness of the volume V and the time $t_2 - t_1$, the integrand should be equal to zero, that is,

$$\gamma(\vec{x}) \frac{\partial u(\vec{x}, t)}{\partial t} = \operatorname{div} \big(D(\vec{x}) \nabla u(\vec{x}, t) \big) + F(\vec{x}, t). \tag{12.1}$$

Equation (12.1) is called the diffusion equation.

If the medium is homogeneous, the quantities $\gamma(\vec{x})$ and $D(\vec{x})$ are constant, and Eq. (12.1) becomes

$$\frac{\partial u(\vec{x}, t)}{\partial t} = a^2 \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \Big) u(\vec{x}, t) + f(\vec{x}, t), \tag{12.2}$$

where

$$a^2 = \frac{D}{\gamma}, \qquad f(\vec{x}, t) = \frac{F(\vec{x}, t)}{\gamma}.$$

When comparing the one-dimensional diffusion equation that follows from (12.2),

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f,$$

with the one-dimensional heat equation (11.6), we can see that they are completely identical in form with the thermal diffusivity coefficient corresponding to the diffusion coefficient at $\gamma = 1$. Moreover, the dimensionalities of these coefficients are identical. For gases, even the numerical values of both coefficients are rather close since the kinetic theory of gases yields an approximate relation $k = D\rho c$, whence $D = k/(\rho c) = a^2$. Thus, in diffusion problems, the amount of the diffusing substance and its concentration play the same roles as the amount of heat and the temperature in heat conduction problems. With the notations of the preceding section, the equation of heat conduction in space

$$c(\vec{x})\rho(\vec{x}) \frac{\partial u(\vec{x}, t)}{\partial t} = \operatorname{div} \big(k(\vec{x}) \nabla u \big) + F(\vec{x}, t) \tag{12.3}$$

follows immediately from Eq. (12.1) upon the change of variables $\gamma(\vec{x}) \to c(\vec{x})\rho(\vec{x})$ and $D(\vec{x}) \to k(\vec{x})$ or can be derived from the heat balance equation in a way similar to the derivation of the diffusion equation from the balance equation for the diffusing substance. Let us consider this derivation.

The degree of heating of a body is characterized by its temperature. Let $u(\vec{x}, t)$ be the temperature at point $\vec{x}$ at time t, $\rho(\vec{x})$ the density of the body material, $k(\vec{x})$ the coefficient of heat conductivity, and $c(\vec{x})$ the specific heat at the point $\vec{x}$. Imagine a surface S_V within the body, encloding a volume V. Assume that there is a heat source $F(\vec{x}, t)$ in the body.

Compose a heat balance equation for this case. Denote by dQ_1 the amount of heat passing through a surface section dS and write

$$dQ_1 = k(\vec{x})\, dS\, dt \frac{\partial u}{\partial n} = k(\vec{x}) dt\, (\nabla u, d\vec{S}). \tag{12.4}$$

Then the amount of heat that will pass through the surface S_V in the time interval $t_2 - t_1$ will be

$$Q_1 = \int\limits_{t_1}^{t_2} dt \oint\limits_{S_V} k(\vec{x})(\nabla u, d\vec{S}) = \int\limits_{t_1}^{t_2} dt \int\limits_{V} \operatorname{div}(k(\vec{x})\nabla u) d\vec{x}. \tag{12.5}$$

The amount of heat that is released or absorbed in the volume V in the time $t_2 - t_1$ is given by

$$Q_2 = \int\limits_{t_1}^{t_2} dt \int\limits_{V} F(\vec{x}, t) d\vec{x}. \tag{12.6}$$

The heat that should be expended for the temperature change $\Delta u = u_2(\vec{x}, t) - u_1(\vec{x}, t)$ in a volume region $dV = d\vec{x}$ in time dt is given by

$$dQ_3 = [u(\vec{x}, t + dt) - u(\vec{x}, t)] c(\vec{x})\rho(\vec{x}) dV = \frac{\partial u}{\partial t} c(\vec{x})\rho(\vec{x}) d\vec{x} dt.$$

The heat going to increase the temperature in the volume V in the time $t_2 - t_1$ is given by

$$Q_3 = \int\limits_{t_1}^{t_2} dt \int\limits_{V} \frac{\partial u}{\partial t} c(\vec{x})\rho(\vec{x}) d\vec{x}. \tag{12.7}$$

Thus, the heat Q_3 expended for heating the volume V in the time $t_2 - t_1$ is equal to the sum of the heat Q_2 produced by the internal sources and the heat Q_1 coming from the external sources

$$Q_3 = Q_1 + Q_2$$

or

$$\int\limits_{t_1}^{t_2} dt \int\limits_{V} \Big[\frac{\partial u}{\partial t} \rho(\vec{x}) c(\vec{x}) - \operatorname{div}(k(\vec{x})\nabla u) - F(\vec{x}, t)\Big] dV = 0.$$

In view of the arbitrariness of the volume V and time $t_2 - t_1$, the integrand should be equal to zero, that is,

$$\rho(\vec{x}) c(\vec{x}) \frac{\partial u}{\partial t} = \operatorname{div}(k(\vec{x})\nabla u) + F(\vec{x}, t). \tag{12.8}$$

♦ Equation (12.8) is called *the three-dimensional heat equation.*

For a homogeneous substance, c, ρ, and k are constants, and Eq. (12.3) takes the form similar to (12.2):

$$u_t = a^2 \Delta u + f(\vec{x}, t). \tag{12.9}$$

This is the heat equation for a homogeneous medium.

Here,

$$a^2 = \frac{k}{c\rho}, \qquad f(\vec{x},t) = \frac{F(\vec{x},t)}{c\rho}$$

and $f(\vec{x},t)$ is called *the source density.*

For solving uniquely the diffusion and heat conduction problems by the above scheme, an initial condition and boundary conditions are specified. The initial condition $u(\vec{x},t)|_{t=0} = \varphi(\vec{x})$ determines the original temperature (concentration) distribution. We shall consider three kinds of boundary conditions:

First kind conditions are specified if a given distribution of the temperature (concentration) u_0 is maintained on the boundary S: $u(\vec{x},t)|_S - u_0 = 0$.

Second kind conditions are specified if a given heat (substance) flux u_1 is maintained on the boundary S:

$$\Big(k\frac{\partial u}{\partial n} + u_1\Big)\Big|_S = 0, \qquad \Big(\Big[D\frac{\partial u}{\partial n} + u_1\Big]\Big|_S = 0\Big).$$

Third kind conditions are specified if a heat (substance) exchange occurs at the boundary S in accordance with Newton's law

$$\Big[k\frac{\partial u}{\partial \vec{n}} + h(u-u_0)\Big]\Big|_S = 0, \qquad \bigg(\Big[D\frac{\partial u}{\partial \vec{n}} + h(u-u_0)\Big]\Big|_S = 0\bigg),$$

where h is the coefficient of heat exchange (penetrability) and u_0 is the temperature of the surrounding medium (concentration of the diffusing substance in the surrounding medium).

13 Transport equations

As follows from the above examples, hyperbolic equations describe, in general, some wave processes (propagation of a wave field). Parabolic equations are more naturally associated with the description of the behavior of some macroscopic characteristics of a system of particles. Actually, if by the quantity $u(\vec{x},t)$, introduced in the preceding section, is meant the concentration of some particles (molecules, ions, etc.), the diffusion equation for a rarefied gas can be obtained by assigning to the particles some mass, so that the diffusion process could be treated as the process of mass transfer by the particles as a result of their redistribution. Similarly, heat conduction can be considered as energy transfer, flow of a viscous fluid as momentum transfer, an electric current as charge transfer, and so on.

To obtain an equation for neutron transport in a nuclear reactor, which is of great practical importance, we should invoke models other than those used in deriving equations of diffusion and heat conduction. Consider neutrons flying in the direction of a unit vector $\vec{n}$. Denote by $N(\vec{x},\vec{n},t)$ the neutron density at point $\vec{x}$ at time t. Assume that all neutrons have the same velocity v and the mean free path of the neutrons is much larger then their dimensions. Denote by Ω a complete group of incompatible events: neutron-neutron collision, A_1; elastic scattering of a neutron by an immobile nucleus (such that the neutron recoils from the nucleus as an elastic ball), A_2; absorption of a neutron by a nucleus, A_3, and division of a nucleus by a neutron with a coefficient of neutron multiplication $k(\vec{x})$, A_4. The distribution of neutrons by directions is uniform (isotropic) both before and after scattering and multiplication. Let the probabilities of the above events be, respectively, $P(A_1) \approx 0$, $P(A_2) = P_2$,

$P(A_3) = P_3$, and $P(A_4) = 1 - P_2 - P_3$, and $F(\vec{x}, \vec{n}, t)$ be the source density. Then the particle flux $U = Nv$ satisfies the one-velocity transport equation with isotropic scattering

$$\frac{1}{v}\frac{\partial u}{\partial t} + (\vec{n}, \operatorname{grad} u) + \frac{\lambda}{l}u = \frac{P_2 + k(1 - P_2 - P_3)}{4\pi l}\int\limits_{S_1} u(\vec{x}, \vec{n}', t)d\vec{n}' + F(\vec{x}, \vec{n}, t), \quad (13.1)$$

where λ is the probability of an event of nuclear decay in a unit time. Equation (13.1) is integro-differential.

In the diffusion approximation, the above process is described by the diffusion equation (12.1), where $u(\vec{x}, t)$ is the neutron concentration in the active zone, D is the coefficient of effective diffusion of neutrons, and the neutron source density $F(\vec{x}, \vec{n}, t)$ is proportional to the neutron density, i.e., $F = \alpha u - \beta u = ku$, where α is the birth factor, β is the absorption factor, and $k = \alpha - \beta$ is the multiplication factor (α and β are determined experimentally).

Note that if $k > 0$, the process of spatial generation of neutrons prevails and the occurrence of a nuclear chain reaction is possible. As a result, Eq. (12.2) (with the coefficient $C = 1$) takes the form

$$u_t - D\Delta u - ku = 0. \quad (13.2)$$

This equation lies at the heart of some mathematical models of nuclear chain reactions.

14 Quantum mechanical equations

The principal statements of quantum mechanics, including the Schrödinger equation, are postulated. The validity of these postulates is verified by checking whether the predictions of quantum mechanics agree with experimental data. Therefore, the Schrödinger equation, in contrast to the equations of diffusion, heat conduction, vibrations, and others cannot be deduced from some more general considerations.

The principal postulates of quantum mechanics are the following:

Postulate 1. In a coordinate representation, the states of a quantum system are described by a normed (ray) vector $\Psi(\vec{x}, t)$ of some Hilbert space $\mathcal{L}$. The function $\Psi(\vec{x}, t)$ is called the state vector and the space $\mathcal{L}$ is called the state space. Each nonzero vector $\Psi(\vec{x}, t)$ is associated with a certain state.

Postulate 2. Every physical quantity A (referred to as observable) of a system is associated with a linear selfadjoint operator $\widehat{A}$ that acts in the Hilbert state space $\mathcal{L}$.

Postulate 3. The evolution of an isolated quantum system is described by the Schrödinger equation

$$-i\hbar\frac{\partial \Psi}{\partial t} + \widehat{\mathcal{H}}(t)\Psi = 0, \quad (14.1)$$

where $\widehat{\mathcal{H}}(t)$ is a linear Hermitian operator which is called the Hamiltonian.

Generally, from physical considerations, the wave function is subject to the following boundary condition

$$\lim_{|\vec{x}|\to\infty} \Psi(\vec{x}, t) = 0,$$

which agrees with Postulate 1.

Postulate 4. When measuring a quantity A, one may obtain only those values of α_n that are eigenvalues of the operator $\widehat{A}$

$$\widehat{A}\Psi_n = \alpha_n \Psi_n. \tag{14.2}$$

If the quantity A has been measured and the measurement result is equal to α_n, then, after measurements, the quantum system is in the state Ψ_{α_n}.

Postulate 5. For the operators of coordinates, $\hat{x}_l$, and momenta, $\hat{p}_j$, the following commutation relations are valid:

$$[\hat{x}_l, \hat{x}_j]_- = [\hat{p}_l, \hat{p}_j]_- = 0, \quad [\hat{x}_l, \hat{p}_j]_- = i\hbar\delta_{lj}, \quad l, j = \overline{1, n}. \tag{14.3}$$

Here $[\widehat{A}, \widehat{B}]_\mp = \widehat{A}\widehat{B} \mp \widehat{B}\widehat{A}$ is the commutator (anticommutator) of the operators $\widehat{A}$ and $\widehat{B}$.

For instance, in the coordinate representation,

$$\hat{x}_l = x_l, \qquad \hat{p}_j = -i\hbar\frac{\partial}{\partial x_j}, \quad l, j = \overline{1, n}.$$

Although the mathematical apparatus of classical mechanics can be derived from that of quantum mechanics (quantum mechanics, in the limit as $\hbar \to 0$, involves classical mechanics), the relations between these physical theories are more intricate and not just the relations between their mathematical schemes. This is a manifestation of the radical difference between quantum mechanics and special relativity which, in the limit as $c \to \infty$, involves nonrelativistic quantum mechanics. This is due to the fact that both relativistic quantum mechanics (special relativity) and nonrelativistic quantum mechanics are formulated in terms of the same language that differs in principal from that of quantum mechanics, and the difficulties involved in their relations are associated with translational and interpretational problems.

Note that the question of the one-to-one correspondence of a physical quantity and a selfadjoint operator still remains open. However, this correspondence is possible (e.g., by Weyl's rule) for quantities which have classical analogs, since classical quantities can be expressed in terms of generalized coordinates and momenta.

Thus, for example, the Hamiltonian function of a classical particle in a potential field has the form

$$\mathcal{H}(\vec{p}, \vec{x}, t) = \frac{\vec{p}^2}{2m} + U(\vec{x}, t), \tag{14.4}$$

where m is the mass of the particle. Make in (14.4) the formal change of variables $x_k \to x_k$, $p_k \to -i\hbar\frac{\partial}{\partial x_k}$ and substitute the result into (14.1) to get

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi - U(\vec{x}, t)\Psi. \tag{14.5}$$

The Schrödinger equation is often referred to as the nonrelativistic wave equation whose solution Ψ is called the wave function. Equation (14.5) enables one, using the values of the wave function at time zero, to find its values at subsequent points in time, thus being an expression of the principle of causality in quantum mechanics.

For a charged particle in an electromagnetic field with potentials $\mathcal{A}_0(\vec{x}, t)$ and $\vec{\mathcal{A}}(\vec{x}, t)$, the Schrödinger equation has the form

$$\Big(i\hbar\frac{\partial}{\partial t} - e\mathcal{A}_0\Big)\Psi = \frac{1}{2m}\Big(- i\hbar\nabla - \frac{e}{c}\vec{\mathcal{A}}\Big)^2\Psi. \tag{14.6}$$

Here c is the velocity of light and e is the electron charge.

To derive the wave equation describing the motion of a relativistic particle, it is necessary to use the relativistic relation between energy and momentum

$$\frac{E^2}{c^2} = \vec{p}^2 + m_0^2 c^2.$$

Here m_0 is the mass of the particle at rest. Making the formal change of variables

$$E \to i\hbar \frac{\partial}{\partial t} \quad \text{and} \quad \vec{p} \to -i\hbar\nabla,$$

we obtain the relativistic wave equation

$$\frac{\hbar^2}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = (\hbar^2 \nabla - m_0^2 c^2)\Psi. \tag{14.7}$$

Equation (14.7) is called the Klein–Gordon equation. Putting here $m_0 = 0$, we obtain the wave equation for a photon. The Schrödinger equation is derived from (14.7) if we put $E_{\rm rel} = E + m_0 c^2$ and assume that $|\vec{p}| \ll m_0 c$. Then the terms nonlinear in v/c can be rejected.

For the case of an external electromagnetic field, the Klein–Gordon equation takes the form

$$\Big(i\hbar\frac{\partial}{\partial t} - e\mathcal{A}_0\Big)^2 \Psi = c^2\Big(-i\hbar\nabla - \frac{e}{c}\vec{\mathcal{A}}\Big)^2 \Psi + m_0^2 c^4 \Psi. \tag{14.8}$$

The question of interpreting the solutions of the Klein–Gordon equation remained open for a long time. The splitting of the energy levels of the hydrogen atom, predicted on the basis of the Klein–Gordon equation, differs from that observed in experiments. Therefore, Eq. (14.7) was believed to have no physical sense. Later it became clear that the Klein–Gordon equation describes spinless particles, such as pi-mesons, which can be both charged (the wave function being complex) and neutral (the wave function being real).

The relativistic wave equation for an electron (particle of spin 1/2), plays a fundamental role in relativistic quantum mechanics and quantum field theory, was proposed by Dirac. This equation has the form

$$\Big\{-i\hbar\partial_t + e\mathcal{A}_0 + c\Big(\vec{\alpha}, \Big[-i\hbar\nabla - \frac{e}{c}\vec{\mathcal{A}}\Big]\Big) - \rho_3 m_0 c^2\Big\}\Psi = 0.$$

Here,

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \qquad \rho_3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

are Dirac matrices in the standard block representation; $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices. The solution of the Dirac equation is a four-dimensional column which is the wave function $\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4)$ of a given quantum system.

15 Maxwell's equations

Assume that in some medium there exists a variable electromagnetic field. Let $\vec{H}(\vec{x},t)$ be the magnetic field strength, $\vec{E}(\vec{x},t)$ the electric field strength, ε the dielectric constant, μ the permeability constant of the medium, $\vec{\jmath}(\vec{x},t)$ the conduction current, and $\rho(\vec{x},t)$ the electric charge density. Then the field strengths satisfy the system of Maxwell's equations

$$\begin{aligned}
&\operatorname{div}(\varepsilon\vec{E}) = 4\pi\rho(\vec{x},t),\\
&\operatorname{div}(\mu\vec{H}) = 0,\\
&\operatorname{rot}\vec{E} = -\frac{1}{c}\frac{\partial(\mu\vec{H})}{\partial t},\\
&\operatorname{rot}\vec{H} = \frac{1}{c}\frac{\partial(\varepsilon\vec{E})}{\partial t} + \frac{4\pi}{c}\vec{\jmath}(\vec{x},t).
\end{aligned} \tag{15.1}$$

In the sequel we shall restrict ourselves to the media with invariable ε and μ. In this case, the vacuum will be associated with $\varepsilon = \mu = 1$.

Let us consider a situation where external currents and charges are absent, that is, $\vec{\jmath}(\vec{x},t) = 0$, $\rho(\vec{x},t) = 0$. Apply the rotor operation to the third and fourth equations to get

$$\begin{aligned}
\operatorname{rot}\operatorname{rot}\vec{E} &= -\frac{\mu}{c}\frac{\partial}{\partial t}(\operatorname{rot}\vec{H}) = -\frac{\mu\varepsilon}{c^2}\frac{\partial^2\vec{E}}{\partial t^2},\\
\operatorname{rot}\operatorname{rot}\vec{H} &= \frac{\varepsilon}{c}\frac{\partial}{\partial t}(\operatorname{rot}\vec{E}) = -\frac{\mu\varepsilon}{c^2}\frac{\partial^2\vec{H}}{\partial t^2}.
\end{aligned}$$

Since $\operatorname{rot}\operatorname{rot}\vec{A} = \operatorname{grad}\operatorname{div}\vec{A} - \Delta\vec{A}$, in view of the first two equations of (15.1), we obtain

$$\begin{aligned}
\Delta\vec{H} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2\vec{H}}{\partial t^2} &= 0,\\
\Delta\vec{E} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} &= 0.
\end{aligned} \tag{15.2}$$

From here it is clear that each Cartesian component of the vectors $\vec{E}(\vec{x},t)$ and $\vec{H}(\vec{x},t)$ satisfies d'Alembert's equation, and the velocity of propagation of electromagnetic waves in a homogeneous dielectric medium is

$$a = \frac{c}{\sqrt{\varepsilon\mu}}.$$

An electromagnetic field can be specified not only by the strengths $\vec{E}(\vec{x},t)$ and $\vec{H}(\vec{x},t)$, but also by electromagnetic potentials. We shall seek the solution of Maxwell's equations (15.1) in the form

$$\begin{aligned}
\vec{H}(\vec{x},t) &= \operatorname{rot}\vec{\mathcal{A}}(\vec{x},t),\\
\vec{E}(\vec{x},t) &= -\operatorname{grad}\mathcal{A}_0(\vec{x},t) - \frac{1}{c}\frac{\partial\vec{\mathcal{A}}(\vec{x},t)}{\partial t},
\end{aligned} \tag{15.3}$$

where $\vec{\mathcal{A}}(\vec{x},t)$ and $\mathcal{A}_0(\vec{x},t)$ are, respectively, the vector and the scalar potentials of the electromagnetic field. Substitute (15.3) into (15.1). The second and third equations become identities, while the other two, for a homogeneous medium, lead to the following equations for the potentials:

$$\begin{aligned}\Delta\mathcal{A}_0 - \frac{\mu\varepsilon}{c^2}\frac{\partial^2\mathcal{A}_0}{\partial t^2} &= -\frac{4\pi}{\varepsilon}\rho(\vec{x},t),\\ \Delta\vec{\mathcal{A}} - \frac{\mu\varepsilon}{c^2}\frac{\partial^2\vec{\mathcal{A}}}{\partial t^2} &= -\frac{4\pi}{c}\vec{\jmath}(\vec{x},t).\end{aligned} \tag{15.4}$$

Here we suppose that the Lorentz gauge condition

$$\frac{\mu\varepsilon}{c^2}\frac{\partial\mathcal{A}_0}{\partial t} + \operatorname{div}\vec{\mathcal{A}} = 0$$

is fulfilled.

16 Stationary physical processes and elliptic equations

In the case of stationary processes such that the external perturbations $[f(\vec{x},t) = f(\vec{x})]$ and the boundary conditions are independent of time $[\alpha(\vec{x},t) = \alpha(\vec{x})$, $\beta(\vec{x},t) = \beta(\vec{x})$, $\mu(\vec{x},t) = \mu(\vec{x})]$, the solution of the heat equation (wave equation) can be sought in the form

$$u(\vec{x},t) = u(\vec{x}).$$

Then we arrive at the equation

$$\operatorname{div}[k(\vec{x})\operatorname{grad}u] - q(\vec{x})u = f(\vec{x}) \tag{16.1}$$

the boundary conditions for which follow from the physical statement of the problem. Let us consider some examples of stationary processes.

16.1 The stationary wave equation

We have already mentioned that wave processes are described by hyperbolic equations. The principal properties of these equations reveal themselves when solving the wave equation (d'Alembert's equation)

$$\Delta u - \frac{1}{a^2(\vec{x})}u_{tt} = 0, \tag{16.2}$$

where $a(\vec{x})$ is the velocity of wave propagation.

In many problems, for instance, in problem of steady-state monochromatic oscillations in an elastic medium, it is natural to seek a solution of Eq. (16.2) in the form

$$u(\vec{x},t) = \mathrm{e}^{-i\omega t}v_\omega(\vec{x}). \tag{16.3}$$

Then the function $v_\omega(\vec{x})$ satisfies the equation

$$\Delta v_\omega + \lambda(\vec{x})v_\omega = 0, \tag{16.4}$$

where

$$\lambda(\vec{x}) = k^2(\vec{x}) = \frac{\omega^2}{a^2(\vec{x})}.$$

The boundary conditions for Eq. (16.4) follow from the boundary conditions for Eq. (16.2), considered earlier.

As a result, we arrive at the problem of finding the eigenvalues ω for which there exist nonzero solutions of Eq. (16.2).

♦ Equation (16.4) is called *Helmholtz' equation.*

If in the wave equation

$$\Delta u - \frac{1}{a^2(\vec{x})} u_{tt} = f(\vec{x}, t) \tag{16.5}$$

the external perturbation $f(\vec{x}, t)$ is periodic with frequency ω and amplitude $g(\vec{x})$

$$f(\vec{x}, t) = \mathrm{e}^{i\omega t} g(\vec{x}),$$

then we can seek a periodic solution $u(\vec{x}, t)$ with the same frequency, but with an unknown amplitude

$$u(\vec{x}, t) = \mathrm{e}^{i\omega t} v(\vec{x}). \tag{16.6}$$

Substitution of (16.6) into (16.5) yields a nonhomogeneous Helmholtz' equation

$$\Delta v + k^2(\vec{x}) = g(\vec{x}), \qquad k(\vec{x}) = \frac{\omega}{a(\vec{x})}. \tag{16.7}$$

♢ In this case, ω is given and we deal with a boundary value problem for Eq. (16.7) rather than with the Sturm–Liouville problem (16.4), where ω is a spectral parameter to be determined.

16.2 Electrostatic and magnetostatic field equations

Let us consider Maxwell's equations (15.1). If the process is stationary, the system of Maxwell's equations breaks up into the two independent systems of equations:

$$\operatorname{div} \varepsilon(\vec{x}) \vec{E} = 4\pi\rho(\vec{x}), \qquad \operatorname{rot} \vec{E} = 0 \tag{16.8}$$

and

$$\operatorname{div} \mu(\vec{x}) \vec{H} = 0, \qquad \operatorname{rot} \vec{H} = \frac{4\pi}{c} \vec{\jmath}. \tag{16.9}$$

♦ Equations (16.8) and (16.9) are called the electrostatic and the magnetostatic field equations, respectively.

Since $\operatorname{rot} \vec{E} = 0$, we may put $\vec{E}(\vec{x}) = -\nabla\Phi(\vec{x})$. Then, if $\varepsilon(\vec{x}) = \text{const}$, we obtain an equation for the electromagnetic potential

$$\Delta\Phi = f(\vec{x}), \qquad f(\vec{x}) = -\frac{4\pi}{\varepsilon} \rho(\vec{x}). \tag{16.10}$$

♦ Equation (16.10) is called *Poisson's equation.*

16.3 The stationary Schrödinger equation

The Schrödinger equation for a quantum system has the form

$$\{-i\hbar\partial_t + \widehat{\mathcal{H}}\}\Psi = 0. \tag{16.11}$$

Assume that the Hamiltonian $\widehat{\mathcal{H}}$ does not depend explicitly on time. Find a solution of Eq. (16.11), $\Psi(\vec{x}, t)$, which would represent a dynamic state with a certain energy E. Seek the function $\Psi(\vec{x}, t)$ in the form

$$\Psi(\vec{x}, t) = e^{-iEt/\hbar} \Psi_E(\vec{x}), \tag{16.12}$$

where $\Psi_E(\vec{x})$ is a function independent of time. Substituting (16.12) into (16.11), we obtain an equation for the function $\Psi_E(\vec{x})$:

$$\widehat{\mathcal{H}}\Psi_E = E\Psi_E. \tag{16.13}$$

♦ Equation (16.13) is called *the stationary Schrödinger equation.*

For a nonrelativistic particle moving in a potential field $U(\vec{x})$, Eq. (16.13) takes the form

$$\Big[-\frac{\hbar^2}{2m}\Delta + U(\vec{x})\Big]\Psi_E = E\Psi_E. \tag{16.14}$$

It is assumed that for the wave function $\Psi(\vec{x})$ the conditions considered in Sec. "Quantum mechanical equations" are fulfilled.

♢ From the mathematical viewpoint, the problem of finding solutions of the nonstationary Schrödinger equation (16.12) or (16.13) is the Sturm–Liouville problem for the eigenvalues and eigenfunctions of the operator $\widehat{\mathcal{H}}$.

16.4 Scattering of particles by an immobile target with a potential of finite action radius

Consider the scattering of a nonrelativistic spinless particle by an immobile particle with a potential of finite action radius. Assume that a directed particle beam of certain energy is incident on a target and is scattered by the target and that the interaction of particles in the beam can be neglected. Then the scattering of an individual beam particle can be considered.

Let the original state of the incident particles is a plane wave $\Psi_k = N\,e^{i(\vec{k},\vec{x})}$. In the general case, it is convenient to consider the original wave packet as a superposition of plane waves:

$$\Psi(\vec{x},0) = \int e^{i(\vec{k},\vec{x})}\Phi(\vec{k})d\vec{k}. \tag{16.15}$$

As a result, we arrive at the equation

$$[\Delta + k^2 - \widetilde{U}(\vec{x})]\Psi = 0, \qquad \widetilde{U}(\vec{x}) = \frac{2m}{\hbar^2}U(\vec{x}), \quad k^2 = \vec{k}^2, \tag{16.16}$$

whose solution for large $|\vec{x}|$ satisfies the condition

$$\Psi(\vec{x}) \underset{|\vec{x}|\to\infty}{\sim} N\,e^{i(\vec{k},\vec{x})} + \frac{e^{i(\vec{k},\vec{x})}}{|\vec{x}|}f_{\vec{k}}\Big(\frac{\vec{x}}{|\vec{x}|}\Big) \tag{16.17}$$

that represents a superposition of the plane wave $e^{i(\vec{k},\vec{x})}$ and a diverging spherical wave. In this case, it is supposed that $\lim\limits_{|\vec{x}|\to\infty} U(\vec{x}) = 0$.

The condition (16.17) can be rewritten in the form

$$\begin{aligned}&\lim_{r\to\infty}|r\,v(\vec{x})| < \infty, \quad r = |\vec{x}|;\\ &\lim_{r\to\infty}\Big\{\frac{\partial v(\vec{x})}{\partial r} - i|\vec{k}|v(\vec{x})\Big\} = 0, \quad v(\vec{x}) = \Psi(\vec{x}) - N\,e^{i(\vec{k},\vec{x})}.\end{aligned} \tag{16.18}$$

♦ Relations (16.18) are called *the Sommerfeld radiation conditions.*

In particular, if there is is an infinite potential barrier $U(\vec{x})$ such that

$$U(\vec{x}) = \begin{cases} \infty, & \vec{x} \in \mathcal{D}; \\ 0, & \vec{x} \notin \mathcal{D}, \end{cases}$$

Eq. (16.16) for the function $v(\vec{x}) = \Psi(\vec{x}) - Ne^{i(\vec{k},\vec{x})}$ transforms into the Helmholtz' equation

$$(\Delta + k^2)v = 0, \qquad \Psi|_S = 0 \tag{16.19}$$

with the Sommerfeld condition (16.18) specified at infinity. Here, S is the boundary of the region $\mathcal{D}$.

16.5 The scalar field

An atomic nucleus is known to consist of nuclons (protons and neutrons) which interact with each other due to nuclear forces. From the viewpoint of quantum field theory, the nuclear forces result from the exchange by mesons between the nucleus nuclons. This type of interaction is referred to as strong interaction of elementary particles. The structure of a nucleus is largely determined by the behavior of the nuclear forces at small distances where they are little known. Therefore, a model approach is widely used in nuclear theory. The models are great in number. They are used to describe the properties of nuclei and nucleus reactions. Among these models there is the scalar field model based on the Klein–Gordon equation

$$\Delta\varphi - \frac{1}{c^2}\varphi_{tt} - k_0^2\varphi = -4\pi\rho(\vec{x}, t), \qquad k_0 = \frac{m_0^2 c^2}{\hbar^2}, \tag{16.20}$$

where, by analogy with Poisson's equation, an additional term in the right side characterizes the field sources φ. The nuclear forces are short-range forces; therefore, the solutions of Eq. (16.20) should rapidly (at least, exponentially) decrease at large distances. Equation (16.20) describes the field of particles with a mass $m_0 = \hbar k_0/c$ at rest, which is produced by a source $\rho(\vec{x}, t)$. For the stationary case, where the source can be considered immobile, we arrive at the Helmholtz' equation for the function $\varphi(\vec{x})$

$$(\Delta - k_0^2)\varphi = f(\vec{x}), \qquad f(\vec{x}) = 4\pi\rho(\vec{x}), \tag{16.21}$$

that differs from Eq. (16.7) only by the sign of the term containing k_0^2.

17 Statement of initial and boundary value problems for equations of mathematical physics

In the foregoing we considered some physical processes whose mathematical description, upon certain physical and geometrical suppositions, is reduced to linear partial differential equations (systems of equations). In this case, different phenomena prove to be described by equations identical in form (e.g., diffusion and heat conduction processes, mechanical vibrations, and electric oscillations). This is due to the fact that these phenomena rest on the fundamental laws of nature.

All equations considered can be assigned to one of the following types:
(a) the wave equation (hyperbolic type)

$$\rho(\vec{x}, t)u_{tt} = \widehat{L}u + f(\vec{x}, t); \tag{17.1}$$

(b) the heat equation (parabolic type)

$$\rho(\vec{x},t)u_t = \widehat{L}u + f(\vec{x},t); \tag{17.2}$$

(c) the equation of steady-state processes (elliptic type)

$$\widehat{L}u + f(\vec{x}) = 0. \tag{17.3}$$

Here,

$$\widehat{L}u = \operatorname{div}(k(\vec{x})\operatorname{grad} u) - q(\vec{x}) = (\nabla, k(\vec{x})\nabla u) - q(\vec{x}) \tag{17.4}$$

and it is assumed that $k(\vec{x}) > 0$, $q(\vec{x}) \geqslant 0$.

We already noted that the general solution of a second order partial differential equation contains, as a rule, two arbitrary functions. From physical considerations it is generally required to find a unique solution. Therefore, a mathematical formulation of a physical problem should involve additional conditions that should be satisfied by the sought-for function on the boundaries of its region of definition.

Let us consider the statement of the most common problems that involve such conditions.

I. The Cauchy problem: Find a function $u(\vec{x},t)$ satisfying, for $t > 0$, Eq. (17.1) or Eq. (17.2) at any point $\vec{x} \in E$ (in this case, the region E coincides with the whole space) and the initial conditions

$$u|_{t=0} = \varphi(\vec{x}), \qquad u_t|_{t=0} = \psi(\vec{x}) \tag{17.5}$$

for Eq. (17.1) and the initial condition

$$u|_{t=0} = \varphi(\vec{x}) \tag{17.6}$$

for Eq. (17.2).

♢ The sought-for function is commonly subject to some additional limitations of general character; for instance, it is required that the condition

$$\lim_{|\vec{x}|\to\infty} |u(\vec{x},t)| \leqslant \infty \tag{17.7}$$

or

$$\lim_{|\vec{x}|\to\infty} |u(\vec{x},t)| = 0 \tag{17.8}$$

be fulfilled at infinity. Conditions like these, as a rule, naturally follow from the physical statement of a problem. The sought-for function can be subject to other limitations; for instance, one can require that

$$\int\limits_E |u(\vec{x},t)|^2\rho(\vec{x})d\vec{x} < \infty \tag{17.9}$$

or

$$\int\limits_E |\operatorname{grad} u(\vec{x},t)|^2\rho(\vec{x})d\vec{x} < \infty,$$

where $\rho(\vec{x}) > 0$, $\vec{x} \in E \subset \mathbb{R}^n$. In this case, the sought-for function can be considered as an element of the Hilbert space $\mathcal{L}$, $u(\vec{x},t) \in \mathcal{L}$, with the norm defined by the relations

$$\|u(\vec{x},t)\|^2 = \int\limits_E |u(\vec{x},t)|^2\rho(\vec{x})d\vec{x} \tag{17.10}$$

or

$$\|u(\vec{x},t)\|^2 = \int\limits_E |\operatorname{grad} u(\vec{x},t)|^2 \rho(\vec{x}) d\vec{x},$$

respectively. Requirements like these arise, in particular, in quantum mechanics. In a more general situation, the function $u(\vec{x},t)$ is considered to be an element of some functional space. In subsequent chapters, we suppose (unless otherwise specified) that condition (17.7) or (17.9) is fulfilled.

II. The mixed problem: Find a function $u(\vec{x},t)$ satisfying, for $t > 0$, $\vec{x} \in E$, Eq. (17.1) [or (17.2)], the initial conditions (17.5) [or (17.6)], and the boundary condition

$$\Big[\alpha(\vec{x})\frac{\partial u}{\partial n} + \beta(\vec{x})u\Big]\Big|_S = \mu(\vec{x},t)\big|_S, \qquad (17.11)$$
$$[\alpha^2(\vec{x}) + \beta^2(\vec{x})]\big|_S \neq 0.$$

Here, S is the boundary of the region E, $\partial u/\partial n$ is the derivative with respect to the outer normal to the surface S. It is assumed that the compatibility condition

$$\Big[\alpha(\vec{x})\frac{\partial \varphi}{\partial n} + \beta(\vec{x})\varphi\Big]\Big|_S = \mu(\vec{x},0)\big|_S \qquad (17.12)$$

is fulfilled.

♦ The boundary conditions (17.11) are referred to as boundary conditions of the first kind (Dirichlet conditions) if $\alpha(\vec{x})|_S \equiv 0$, of the second kind (Neumann conditions) if $\beta(\vec{x})|_S \equiv 0$, and of the third kind (Robin conditions) otherwise.

III. The boundary value problem (problem of determining a stationary regime): Find the function $u(\vec{x},t)$ satisfying, in the region E, Eq. (17.1) [or (17.2)] and the boundary condition (17.11) (without initial conditions).

For elliptic equations: Find a function satisfying, in the region E ($\vec{x} \in E$), Eq. (17.3) and the boundary conditions

$$\Big[\alpha(\vec{x})\frac{\partial u}{\partial n} + \beta(\vec{x})u\Big]\Big|_{S_E} = \mu(\vec{x})|_{S_E}, \quad [\alpha^2(\vec{x}) + \beta^2(\vec{x})]\big|_{S_E} \neq 0. \qquad (17.13)$$

♦ The boundary value problem with $\alpha(\vec{x})|_S \equiv 0$ is called the Dirichlet problem and with $\beta(\vec{x}) \equiv 0$ the Neumann problem.

To this point we have considered internal problems with $\vec{x} \in E$.

♢ If it is necessary to find a solution $u(\vec{x},t)$ satisfying conditions (17.11) and (17.13) in an infinite region E^* external to the surface S, it is demanded that the regularity condition (17.7) or (17.8) be fulfilled at infinity. Such a problem is called an external boundary value problem.

IV. The Sturm–Liouville problem. In the section devoted to special functions we considered the Sturm–Liouville problem for ordinary differential equations. This problem can be naturally generalized to partial differential equations.

Consider a partial differential equation

$$\widehat{L}v + \lambda\rho(\vec{x})v = 0, \qquad \vec{x} \in E \qquad (17.14)$$

with a homogeneous boundary condition

$$\Big(\alpha\frac{\partial v}{\partial n} + \beta v\Big)\Big|_{S_E} = 0. \qquad (17.15)$$

Here,

$$\widehat{L}v = \operatorname{div}\big(k(\vec{x})\nabla v\big) - q(\vec{x})v. \tag{17.16}$$

♦ The problem of finding the values of the parameter λ for which there exists a nontrivial solution of Eq. (17.14) with the boundary conditions (17.15) is called *the Sturm–Liouville problem*. The values of the parameter λ for which there exists a solution of the Sturm–Liouville problem are called *eigenvalues* and the related functions $v_\lambda(\vec{x})$ are called *eigenfunctions*.

♦ The totality of all eigenvalues $\{\lambda_n\}$ of the Sturm–Liouville problem is called *a spectrum and the totality of all eigenvalues* and related eigenfunctions $[\lambda_n, v_n(\vec{x})]$, $n = \overline{1,\infty}$ is called *a spectral series*.

♢ In the following sections, we shall focus on the methods of solving problems of these types.

V. Classical and generalized solutions. We shall assume, as a rule, that in all cases under consideration, it is required to find a solution $u(\vec{x}, t)$ which would be continuous together with its partial derivatives of the corresponding order (e.g., for a second order equation, the solution is continuous together with its second order partial derivatives with respect to all variables). Such solutions are called classical and the statement of the corresponding boundary value problem in this class of functions is called a classical setting. However, in some interesting cases, initial and/or boundary conditions can be specified by nonsmooth (discontinuous) functions, and the solutions of problems of this type are not smooth functions. In particular, it is such problems that have led to the notion of a generalized solution.

Consider a linear second order partial differential equation

$$\widehat{L}u = f(\vec{x}), \qquad \widehat{L}u = \big(\nabla, k(\vec{x})\nabla u\big) - q(\vec{x})u, \quad \vec{x} \in \mathbb{R}^n, \tag{17.17}$$

where $k(\vec{x})$ and $q(\vec{x})$ are smooth functions.

♦ By *generalized solution* of Eq. (17.17) in a region $E \subset \mathbb{R}^n$ is meant a generalized function $u(\vec{x})$ satisfying in this region Eq. (17.17) in the generalized sense, that is,

$$\langle \widehat{L}u(\vec{x}) | \varphi(\vec{x}) \rangle = \langle f(\vec{x}) | \varphi(\vec{x}) \rangle, \tag{17.18}$$

where $\varphi(\vec{x})$, $\vec{x} \in E \subset \mathbb{R}^n$, belongs to the space of basic functions $\mathcal{D}(E)$ [to the Schwartz space $\mathcal{S}(E)$ for unbounded regions (see Sec. “Basic and generalized functions” of Part II)].

♢ Any classical solution is a generalized one as well.

Consider several statements of initial and boundary value problems that lead to generalized functions.

VI. Fundamental solutions. The notion of a fundamental solution, introduced for a linear operator of one variable (see Sec. “Fundamental solutions of linear differential operators” of Part II) is naturally generalized to the case of several variables.

♦ By *the fundamental solution* (*influence function*) of the partial differential equation $\widehat{L}u(\vec{x}) = 0$ or the fundamental solution of the linear differential operator $\widehat{L}$ (17.17) is meant a generalized function $\mathcal{E}(\vec{x}, \vec{y})$ which, for every fixed $\vec{y} \in \mathbb{R}^n$, satisfies in $\mathbb{R}^n$ the equation

$$\widehat{L}_x \mathcal{E}(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y}), \tag{17.19}$$

where, by definition, $\delta(\vec{x} - \vec{y}) = \delta(x_1 - y_1)\delta(x_2 - y_2)\ldots\delta(x_n - y_n)$.

The fundamental solution $\mathcal{E}(\vec{x},\vec{y})$ makes it possible to find a particular solution of the nonhomogeneous equation (17.17) by the formula

$$u(\vec{x}) = \int\limits_E \mathcal{E}(\vec{x},\vec{y})f(\vec{y})d\vec{y} \tag{17.20}$$

on the assumption that the integral in the right side exists. The validity of relation (17.20) can be checked by acting on both sides of this formula with the operator $\widehat{L}$ to get

$$\widehat{L}u(\vec{x}) = \int\limits_E \widehat{L}_x\mathcal{E}(\vec{x},\vec{y})f(\vec{y})d\vec{y} = \int\limits_E \delta(\vec{x}-\vec{y})f(\vec{y})d\vec{y} = f(\vec{x}).$$

$\Diamond$ If the operator $\widehat{L}$ is a differential operator with constant coefficients, then by the fundamental solution of the operator $\widehat{L}$ is often meant a generalized function $\mathcal{E}(\vec{x})$ which is a solution of the equation

$$\widehat{L}\mathcal{E}(\vec{x}) = \delta(\vec{x}). \tag{17.21}$$

Obviously, these fundamental solutions are related as

$$\mathcal{E}(\vec{x},\vec{y}) = \mathcal{E}(\vec{x}-\vec{y}). \tag{17.22}$$

In this case, if the function $f(\vec{x})$ is such that the convolution $f*g$ (see Sec. "Convolution of generalized functions" of Part II) exists, relation (17.20) takes the form

$$u(\vec{x}) = f*g = \int g(\vec{x}-\vec{y})f(\vec{y})d\vec{y}. \tag{17.23}$$

The solution of the form of (17.23) is sometimes referred to as a solution in a potential form.

Its other name, the influence function, becomes clear if the nonhomogeneous term $f(\vec{x})$ in Eq. (17.17) is represented as the "sum" of point sources $f(\vec{y})\delta(\vec{x}-\vec{y})$, that is,

$$f(\vec{x}) = \int\limits_E \delta(\vec{x}-\vec{y})f(\vec{y})d\vec{y}.$$

By virtue of (17.19) and (17.20), each point source $f(\vec{y})\delta(\vec{x}-\vec{y})$ affects an object placed at a point $\vec{x}$ in accordance with formula $f(\vec{y})\mathcal{E}(\vec{x},\vec{y})$. Therefore, the solution

$$u(\vec{x}) = \int\limits_E \mathcal{E}(\vec{x},\vec{y})f(\vec{y})d\vec{y}$$

is a superposition of these effects.

$\Diamond$ It can be shown that a fundamental solution exists for any differential equation with constant coefficients and for any elliptic differential equations (see [19]).

Below we shall present fundamental solutions for the operators frequently used in mathematical physics.

(a) Laplacian operator $\Delta_n = \sum\limits_{k=1}^{n}\dfrac{\partial^2}{\partial x_k^2}$:

$$\Delta_n\mathcal{E}_n(\vec{x},\vec{y}) = \delta(\vec{x}-\vec{y}),$$

$$\mathcal{E}_n(\vec{x},\vec{y}) = -\frac{1}{(\delta_{2n}+n-2)\sigma_n}e_n(\vec{x}-\vec{y}).$$

Here, δ_{2n} is Kronecker's delta symbol and σ_n is the surface area of a unit sphere in $\mathbb{R}^n$,

$$\sigma_n = \int\limits_{|\vec{x}|=1} dS = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \qquad \sigma_2 = 2\pi, \qquad \sigma_3 = 4\pi$$

and

$$e_n(\vec{x}) = \begin{cases} \ln \dfrac{1}{|\vec{x}|}, & n = 2, \\ \dfrac{1}{|\vec{x}|^{n-2}}, & n \geqslant 2. \end{cases}$$

(b) Helmholtz' operator $\Delta_n + k^2$:

$$(\Delta_n + k^2)\mathcal{E}_n(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y}),$$

$$\mathcal{E}_n(\vec{x}, \vec{y}) = \begin{cases} \pm\dfrac{i}{4} H_0^{(1,2)}(k|\vec{x} - \vec{y}|), & n = 2; \\ -\dfrac{e^{\pm k|\vec{x}-\vec{y}|}}{4\pi|\vec{x} - \vec{y}|}, & n = 3. \end{cases}$$

Here $H_0^{(1)}$ and $H_0^{(2)}$ are the Hankel functions of order zero; k is, in the general case, a complex variable.

(c) Heat conduction operator $\dfrac{\partial}{\partial t} - a^2\Delta_n$:

$$\Big(\frac{\partial}{\partial t} - a^2\Delta_n\Big)\mathcal{E}_n(\vec{x}, \vec{y}, t) = \delta(t)\delta(\vec{x} - \vec{y}), \qquad \vec{x}, \vec{y} \in \mathbb{R}^n, \quad t \in \mathbb{R}^1,$$

$$\mathcal{E}_n(\vec{x}, \vec{y}, t) = \frac{\theta(t)}{(2a\sqrt{\pi t})^n} \exp\Big(-\frac{|\vec{x} - \vec{y}|^2}{4a^2 t}\Big).$$

(d) Wave operator (d'Alembert operator) $\Box_n = \dfrac{\partial^2}{\partial t^2} - a^2\Delta_n$:

$$\Box_n\mathcal{E}_n(\vec{x}, \vec{y}, t) = \delta(t)\delta(\vec{x} - \vec{y}),$$

$$\mathcal{E}_n(\vec{x}, \vec{y}, t) = \begin{cases} \dfrac{1}{2a}\theta(at - |\vec{x} - \vec{y}|), & n = 1; \\ \dfrac{\theta(at - |\vec{x} - \vec{y}|)}{2\pi a\sqrt{a^2t^2 - |\vec{x} - \vec{y}|^2}}, & n = 2; \\ \dfrac{\theta(t)\delta(a^2t^2 - |\vec{x} - \vec{y}|^2)}{2\pi a}, & n \geqslant 3. \end{cases}$$

The fundamental solution $\mathcal{E}(\vec{x}, \vec{y})$ of the operator $\widehat{L}$ is not unique and is determined up to a term $g_0(\vec{x}, \vec{y})$ which, for each fixed $\vec{y}$, is an arbitrary solution of the homogeneous equation

$$\widehat{L}g_0(\vec{x}, \vec{y}) = 0.$$

This makes it possible to find fundamental solutions that satisfy some additional conditions, including the solutions of Eqs. (17.19) with boundary, initial, and mixed (initial plus boundary) conditions. The fundamental solutions of the corresponding problems are also called Green's functions for these problems. For Green's functions, we shall use the notations $g(\vec{x}, \vec{y})$ and $G(\vec{x}, \vec{y})$.

VII. The generalized Cauchy problem

♦ By the generalized Cauchy problem for Eq. (17.1) or Eq. (17.2) is meant the problem of finding a generalized function $\tilde{u}(\vec{x},t)$ that vanishes at $t<0$ and satisfies the wave equation

$$\rho(\vec{x})\tilde{u}_{tt} = \widehat{L}\tilde{u} + F(\vec{x},t) \qquad (17.24)$$

or

$$\rho(\vec{x})\tilde{u}_{t} = L\tilde{u} + F(\vec{x},t). \qquad (17.25)$$

It can be shown that the solutions of the classical Cauchy problem (17.1), (17.5) and that of the generalized Cauchy problem (17.24) are related as (see [19])

$$\begin{gathered} \tilde{u}(\vec{x},t) = \theta(t)u(\vec{x},t), \\ F(\vec{x},t) = f(\vec{x},t) + \varphi(\vec{x})\delta'(t) + \psi(\vec{x})\delta(t). \end{gathered} \qquad (17.26)$$

Similarly, for the Cauchy problem (17.1), (17.6) we have

$$\tilde{u}(\vec{x},t) = \theta(t)u(\vec{x},t), \qquad F(\vec{x},t) = f(\vec{x},t) + \varphi(\vec{x})\delta(t). \qquad (17.27)$$

CHAPTER 4
The Sturm–Liouville problem for partial differential equations

18 Statement of the problem

In the section devoted to special functions, we considered the Sturm–Liouville problem for ordinary differential equations (see Sec. III.2). The notion of a Sturm–Liouville problem can be generalized in a natural way to partial differential equations.

Let E be a bounded region in a space $\mathbb{R}^n$ and S_E is its boundary (smooth surface). Consider a partial differential equation

$$\widehat{L}v + \lambda\rho(\vec{x})v = 0, \qquad \vec{x} \in E \tag{18.1}$$

with a homogeneous boundary condition

$$\Big(\alpha(\vec{x})\frac{\partial v}{\partial n} + \beta(\vec{x})v\Big)\Big|_{S_E} = 0. \tag{18.2}$$

Here,

$$\widehat{L}v = \operatorname{div}\big(k(\vec{x})\operatorname{grad} v\big) - q(\vec{x})v = \big(\nabla, k(\vec{x})\nabla v\big) - q(\vec{x})v, \tag{18.3}$$

$\partial v/\partial n$ is the derivative with respect the inner normal to the surface S_E, and $\rho(\vec{x})$ is a specified function of positive sign.

♦ The problem of finding the values of the parameter λ at which a nontrivial solution $v_\lambda(\vec{x})$ exists for Eq. (18.1) with boundary value conditions (18.2) is called the Sturm–Liouville problem. The values of the parameter λ for which there exists a solution of the Sturm–Liouville problem are called eigenvalues and the corresponding functions $v_\lambda(\vec{x})$ are called eigenfunctions.

♦ The totality of all eigenfunctions $\{\lambda_n\}$ of a Sturm–Liouville problem is called a spectrum and the totality of the eigenfunctions and the corresponding eigenfunctions $[\lambda_n, v_n(\vec{x})]$, $n = \overline{0, \infty}$ is called a spectral series.

Let us list the principal properties of the eigenvalues and eigenfunctions of the Sturm–Liouville problem for partial differential equations.

Property 1. There exists an infinite denumerable set of eigenvalues $\{\lambda_n\}$ and eigenfunctions $\{v_n(\vec{x})\}$, and the numeration of the eigenvalues λ_n can be chosen so that they will unlimitedly increase as the number n infinitely increases. Every eigenvalue is associated with a finite number of linearly independent eigenfunctions.

♢ In contrast to the Sturm–Liouville problem for ordinary differential equations, the eigenvalues for partial differential equations can be degenerate, that is, an eigenvalue can be associated with several linearly independent eigenfunctions. The number of these eigenfunctions is called the multiplicity of the eigenvalue or the multiplicity of degeneration. Below we assume that each eigenvalue is present in the spectrum of a Sturm–Liouville problem as many times as its multiplicity is.

Property 2. For $q(\vec{x}) \geqslant 0$ and $\alpha = 0$, $\beta = 1$ (Dirichlet boundary condition), the eigenvalues of a Sturm–Liouville problem are positive:

$$\lambda_n > 0, \qquad n = \overline{0, \infty}.$$

Property 3. The eigenfunctions of the Sturm–Liouville problem (18.1), (18.2) satisfy the orthogonality relationship

$$\langle v_n(\vec{x})|v_m(\vec{x})\rangle_\rho = \|v_n(\vec{x})\|^2\delta_{nm}, \qquad n,m=\overline{0,\infty}, \tag{18.4}$$

where

$$\langle v|u\rangle_\rho = \langle v(\vec{x})|u(\vec{x})\rangle_\rho = \int\limits_E v(\vec{x})u(\vec{x})\rho(\vec{x})d\vec{x}, \tag{18.5}$$

$$\|v\| = \|v(\vec{x})\| = \sqrt{\langle v(\vec{x})|v(\vec{x})\rangle_\rho}. \tag{18.6}$$

◊ In what follows we assume that the eigenfunctions corresponding to a degenerate eigenvalue are chosen to be orthogonal. This can always be done by, e.g., the Schmidt orthogonalization method (see Sec. "Classical orthogonal polynomials" of Part III).

Property 4 (Steklov's expansion theorem). If a function $f(\vec{x})$ is twice continuously differentiable in a closed region $\bar{E}$ and satisfies the boundary condition (18.2), it can be expanded in an absolutely and uniformly converging series in terms of the eigenfunctions of the Sturm–Liouville problem (18.1) and (18.2)

$$f(\vec{x}) = \sum_{n=0}^{\infty} C_n v_n(\vec{x}), \tag{18.7}$$

where

$$C_n = \frac{1}{\|v_n\|^2}\langle f(\vec{x})|v_n(\vec{x})\rangle_\rho. \tag{18.8}$$

The series (18.7) is called the Fourier series of the function $f(\vec{x})$ in orthogonal functions $\{v_n(\vec{x})\}$, $n=\overline{0,\infty}$. The coefficients (18.8) are called the Fourier coefficients.

Corollary. The system of eigenfunctions $\{v_n(\vec{x})\}$ of a Sturm–Liouville problem satisfies the completeness condition

$$\sum_{n=0}^{\infty} \frac{1}{\|v_n\|^2}\rho(\vec{y})v_n(\vec{x})v_n(\vec{y}) = \delta(\vec{x}-\vec{y}) \tag{18.9}$$

in the class of functions twice continuously differentiable in a region $\bar{E}$ for which the homogeneous boundary condition (18.2) is fulfilled.

Proof. Substitute (18.8) into (18.7) and interchange the summation and the integration to get

$$\int\limits_E f(\vec{y})\sum_{n=0}^{\infty}\frac{1}{\|v_n\|^2}\rho(\vec{y})v_n(\vec{x})v_n(\vec{y})\,d\vec{y} = f(\vec{x})$$

from which, in accordance with the definition of the delta function, follows (18.9).

◊ The proofs of properties 1–4 are completely identical to those of the corresponding properties of the Sturm–Liouville problem for ordinary differential equations.

19 The Sturm–Liouville problem and initial boundary value problems for equations of mathematical physics

19.1 Reduction of the general problem

Let us consider the nonhomogeneous equation

$$\rho(\vec{x})\widehat{P}_t u = \widehat{L}u + f(\vec{x}, t), \qquad \vec{x} \in E, \tag{19.1}$$

where

$$\widehat{P}_t u = \sum_{j=0}^{2} a_j(t) \frac{\partial^j u}{\partial t^j}, \tag{19.2}$$

and the operator $\widehat{L}$ is given by formula (18.3). Equation (19.2) is hyperbolic if $a_2 \neq 0$ and parabolic if $a_2 = 0$ and $a_1 \neq 0$.

In particular, for

$$a_2 = \frac{1}{a^2}, \quad a_0 = a_1 = 0, \quad \text{and} \quad \widehat{P}_t u = \frac{1}{a^2} u_{tt}$$

Eq. (19.2) becomes a wave equation, and, if additionally $\widehat{L} = \Delta$, it becomes d'Alembert's equation

$$\frac{1}{a^2} u_{tt} = \Delta u.$$

If in Eq. (19.2)

$$a_0 = a_2 = 0, \quad a_1 = \frac{1}{a^2}, \quad \text{and} \quad \widehat{P}_t u = \frac{1}{a^2} u_t,$$

it becomes the heat equation. In what follows, we shall be interested in the case where $\widehat{L} = \Delta$, and the heat equation will take the form

$$\frac{1}{a^2} u_t = \Delta u.$$

Formulate for Eq. (19.1) the initial conditions

$$u|_{t=0} = \varphi(\vec{x}), \qquad u_t|_{t=0} = \psi(\vec{x}) \tag{19.3}$$

and the boundary conditions

$$\Big(\alpha(\vec{x}) \frac{\partial u}{\partial n} + \beta(\vec{x}) u\Big)\Big|_{S_E} = \mu(\vec{x}, t)\big|_{S_E}, \quad |\alpha(\vec{x})| + |\beta(\vec{x})| \neq 0. \tag{19.4}$$

◊ For the heat equation, the initial condition has the form

$$u|_{t=0} = \varphi(\vec{x}). \tag{19.5}$$

♦ By the classical solution of the problem (19.1)–(19.5) is meant a function $u(\vec{x}, t)$ defined and continuous together with its derivatives, the second ones inclusive, in a region E at $t \in [0, T]$ and satisfying the boundary condition (19.4) and the initial conditions (19.3) or (19.5).

Direct checking shows that the following is true:

Statement 19.1. *Let functions $u_1(\vec{x},t)$, $u_2(\vec{x},t)$, and $u_3(\vec{x},t)$ be the classical solutions of the following problems:*

$$\begin{cases} \rho(\vec{x})\widehat{P}_t u_1 = \widehat{L}u_1, \qquad \vec{x}\in E, \\ \Big(\alpha\dfrac{\partial u_1}{\partial n} + \beta u_1\Big)\Big|_S = 0, \quad u_1|_{t=0} = \varphi(\vec{x}), \ \dfrac{\partial u_1}{\partial t}\Big|_{t=0} = \psi(\vec{x}); \end{cases} \tag{19.6}$$

$$\begin{cases} \rho(\vec{x})\widehat{P}_t u_2 = \widehat{L}u_2 + f(\vec{x},t), \qquad \vec{x}\in E, \\ \Big(\alpha\dfrac{\partial u_2}{\partial n} + \beta u_2\Big)\Big|_S = 0, \quad u_2|_{t=0} = \dfrac{\partial u_2}{\partial t}\Big|_{t=0} = 0; \end{cases} \tag{19.7}$$

$$\begin{cases} \rho(\vec{x})\widehat{P}_t u_3 = \widehat{L}u_3, \qquad \vec{x}\in E, \\ \Big(\alpha\dfrac{\partial u_3}{\partial n} + \beta u_3\Big)\Big|_S = \mu(\vec{x},t)|_S, \quad u_3|_{t=0} = \dfrac{\partial u_3}{\partial t}\Big|_{t=0} = 0, \end{cases} \tag{19.8}$$

then the solution $u(\vec{x},t)$ of the problem (19.1), (19.3), *and* (19.4) *has the form*

$$u(\vec{x},t) = u_1(\vec{x},t) + u_2(\vec{x},t) + u_3(\vec{x},t). \tag{19.9}$$

The procedure that converts the initial boundary value problem (19.1), (19.3), *and* (19.4) *to the simpler problems* (19.6)–(19.8) *is called* reducting the general problem*.*

◊ The problem (19.6) is a mixed problem with a homogeneous boundary condition and a nonhomogeneous initial condition for a second order linear partial differential equation; the problem (19.7) is a mixed problem with homogeneous boundary and initial conditions for a nonhomogeneous linear equation, and the problem (19.8) is a mixed problem with nonhomogeneous boundary conditions and homogeneous initial conditions for a homogeneous linear equation.

19.2 Nonhomogeneous initial conditions

Let us consider the problem (19.6). We shall seek its solution in the form

$$u_1(\vec{x},t) = \sum_{n=0}^{\infty} T_n(t)v_n(\vec{x}), \tag{19.10}$$

where $v_n(\vec{x})$ is a solution of the Sturm–Liouville problem (18.1), (18.2).

Substitute (19.10) into (19.6) to get

$$\rho(\vec{x})\sum_{n=0}^{\infty} v_n(\vec{x})\widehat{P}_t T_n(t) = \sum_{n=0}^{\infty} \widehat{L}T_n(t)v_n(\vec{x}) = \rho(\vec{x})\sum_{n=0}^{\infty}(-\lambda_n)T_n(t)v_n(\vec{x}).$$

Equating the coefficients of identical functions $v_n(\vec{x})$, we obtain for the determination of the functions $T_n(t)$ the following Cauchy problem:

$$\widehat{P}_t T_n + \lambda_n T_n = 0, \qquad T_n(0) = \varphi_n, \quad T_n'(0) = \psi_n, \tag{19.11}$$

where φ_n and ψ_n are coefficients (18.8) of the expansions of the functions $\varphi(\vec{x})$ and $\psi(\vec{x})$ in Fourier series in functions $v_n(\vec{x})$

$$\varphi(\vec{x}) = \sum_{n=0}^{\infty} \varphi_n v_n(\vec{x}), \qquad \psi(\vec{x}) = \sum_{n=0}^{\infty} \psi_n v_n(\vec{x}); \tag{19.12}$$

$$\varphi_n = \frac{1}{\|v_n\|^2}\langle\varphi(\vec{x})|v_n(\vec{x})\rangle_\rho, \quad \psi_n = \frac{1}{\|v_n\|^2}\langle\psi(\vec{x})|v_n(\vec{x})\rangle_\rho. \tag{19.13}$$

In particular, the Cauchy problem (19.11) for the heat equation takes the form

$$\frac{1}{a^2}T_n' + \lambda_n T_n = 0, \qquad T_n(0) = \varphi_n, \tag{19.14}$$

and its solution is given by the functions

$$T_n(t) = \varphi_n e^{-\lambda_n a^2 t}. \tag{19.15}$$

Similarly, the Cauchy problem for the wave equation

$$\frac{1}{a^2}T_n'' + \lambda_n T_n = 0, \qquad T_n(0) = \varphi_n, \quad T_n'(0) = \psi_n, \tag{19.16}$$

will have the solution

$$T_n(t) = \varphi_n \cos(a\sqrt{\lambda_n}t) + \frac{\psi_n}{a\sqrt{\lambda_n}}\sin(a\sqrt{\lambda_n}t). \tag{19.17}$$

Thus, the solution of the mixed problem (19.6) for the heat equation has the form

$$u_1(\vec{x}, t) = \sum_{n=0}^{\infty}\varphi_n e^{-\lambda_n a^2 t}v_n(\vec{x}), \tag{19.18}$$

and for the wave equation it is

$$u_1(\vec{x}, t) = \sum_{n=0}^{\infty}\Big[\varphi_n \cos(a\sqrt{\lambda_n}t) + \frac{\psi_n}{a\sqrt{\lambda_n}}\sin(a\sqrt{\lambda_n}t)\Big]v_n(\vec{x}). \tag{19.19}$$

♦ The function $g(\vec{x}, \vec{y}, t)$ is called *Green's function or the fundamental solution of the heat equation* if, for any fixed $\vec{y} \in E$, the following relations are valid:

$$\rho(\vec{x})g_t(\vec{x}, \vec{y}, t) = \widehat{L}_x g(\vec{x}, \vec{y}, t), \quad \vec{x} \in E; \tag{19.20}$$

$$\Big[\alpha(\vec{x})\frac{\partial g(\vec{x}, \vec{y}, t)}{\partial n_x} + \beta(\vec{x})g(\vec{x}, \vec{y}, t)\Big\}\Big|_{S_E} = 0; \tag{19.21}$$

$$g|_{t=0} = \delta(\vec{x} - \vec{y}), \quad |\alpha(\vec{x})| + |\beta(\vec{x})| \neq 0. \tag{19.22}$$

If the region E coincides with the whole space, a function $g(\vec{x}, \vec{y}, t)$ satisfying conditions (19.20), (19.22) is called the Green's function of the Cauchy problem for Eq. (19.6) and is denoted by $G(\vec{x}, \vec{y}, t)$.

Substituting (19.22) into (19.13) and the result into (19.18), we arrive at the following statements:

Statement 19.2. *The Green's function of a mixed problem for the heat equation* (19.1) $\widehat{P}_t = \frac{1}{a^2}\partial_t$ *can be represented in the form*

$$g(\vec{x}, \vec{y}, t) = \sum_{n=0}^{\infty}\frac{1}{\|v_n\|^2}\rho(\vec{y})e^{-\lambda_n a^2 t}v_n(\vec{x})v_n(\vec{y}). \tag{19.23}$$

Statement 19.3. *Let $g(\vec{x},\vec{y},t)$ be the Green's function of a mixed problem for the heat equation. Then the solution of the problem* (19.6) *has the form*

$$u_1(\vec{x},t) = \int\limits_E g(\vec{x},\vec{y},t)\varphi(\vec{y})d\vec{y}. \tag{19.24}$$

Actually, substituting the Fourier coefficients (19.13) into relation (19.18), in view of the explicit form of Green's function (19.23), we arrive at (19.24).

♦ The functions $g(\vec{x},\vec{y},t)$ and $\mathfrak{g}(\vec{x},\vec{y},t)$ are called the Green's functions of the mixed problem for the wave equation if for any fixed $\vec{y}\in E$ the following conditions are valid:

$$\rho(\vec{x})g_{tt}(\vec{x},\vec{y},t) = \widehat{L}_x g(\vec{x},\vec{y},t), \quad \vec{x}\in E; \tag{19.25}$$

$$\Big[\alpha(\vec{x})\frac{\partial g(\vec{x},\vec{y},t)}{\partial n_x} + \beta(\vec{x})g(\vec{x},\vec{y},t)\Big\}\Big|_{S_E} = 0; \tag{19.26}$$

$$g|_{t=0} = 0, \qquad g_t|_{t=0} = \delta(\vec{x}-\vec{y}). \tag{19.27}$$

$$\rho(\vec{x})\mathfrak{g}_{tt}(\vec{x},\vec{y},t) = \widehat{L}_x \mathfrak{g}(\vec{x},\vec{y},t), \quad \vec{x}\in E; \tag{19.28}$$

$$\Big[\alpha(\vec{x})\frac{\partial \mathfrak{g}(\vec{x},\vec{y},t)}{\partial n_x} + \beta(\vec{x})\mathfrak{g}(\vec{x},\vec{y},t)\Big\}\Big|_{S_E} = 0; \tag{19.29}$$

$$\mathfrak{g}|_{t=0} = \delta(\vec{x}-\vec{y}), \qquad \mathfrak{g}_t|_{t=0} = 0. \tag{19.30}$$

When the region E coincides with the whole space, functions $g(\vec{x},\vec{y},t)$ and $\mathfrak{g}(\vec{x},\vec{y},t)$ satisfying conditions (19.27)–(19.30) are called the Green's functions of the Cauchy problem for the wave equation (19.1) $\widehat{P}_t = \frac{1}{a^2}\partial_t^2$ and are denoted by $G(\vec{x},\vec{y},t)$ and $\mathfrak{G}(\vec{x},\vec{y},t)$.

Similarly, for the wave equation we arrive at the following statement:

Statement 19.4. *The Green's functions of a mixed problem for the wave equation $\widehat{P}_t = \frac{1}{a^2}\partial_t^2$* (19.1) *can be represented as*

$$g(\vec{x},\vec{y},t) = \sum_{n=0}^{\infty}\frac{1}{a\sqrt{\lambda_n}}\frac{1}{\|v_n\|^2}\rho(\vec{y})\sin(a\sqrt{\lambda_n}t)v_n(\vec{x})v_n(\vec{y}); \tag{19.31}$$

$$\mathfrak{g}(\vec{x},\vec{y},t) = \sum_{n=0}^{\infty}\frac{1}{\|v_n\|^2}\rho(\vec{y})\cos(a\sqrt{\lambda_n}t)v_n(\vec{x})v_n(\vec{y}). \tag{19.32}$$

Substituting the Fourier coefficients (19.13) into relation (19.19), in view of the explicit form of Green's functions (19.31) and (19.32), we make sure of the validity of the following statement:

Statement 19.5. *Let $g(\vec{x},\vec{y},t)$ and $\mathfrak{g}(\vec{x},\vec{y},t)$ be the Green's functions of a mixed problem for the wave equation. Then the solution of the problem* (19.6) *can be represented in the form*

$$u_1(\vec{x},t) = \int\limits_E g(\vec{x},\vec{y},t)\psi(\vec{y})d\vec{y} + \int\limits_E \mathfrak{g}(\vec{x},\vec{y},t)\varphi(\vec{y})d\vec{y}. \tag{19.33}$$

◊ If $\widehat{P}_t = \frac{1}{a^2}\partial_t^2$ is a wave operator, then from (19.31) and (19.32) it follows that

$$\mathfrak{g}(\vec{x},\vec{y},t) = g_t(\vec{x},\vec{y},t). \tag{19.34}$$

19.3 The linear nonhomogeneous equation

Let us now consider the problem (19.7). We shall seek its solution in the form

$$u_2(\vec{x}, t) = \sum_{n=0}^{\infty} \Theta_n(t) v_n(\vec{x}). \tag{19.35}$$

Substitution of (19.35) into (19.7) yields

$$\rho(\vec{x}) \sum_{n=0}^{\infty} v_n(\vec{x}) \widehat{P}_t T_n(t) = \sum_{n=0}^{\infty} \widehat{L} T_n(t) v_n(\vec{x}) + f(\vec{x}, t) =$$
$$= \rho(\vec{x}) \sum_{n=0}^{\infty} (-\lambda_n) T_n(t) v_n(\vec{x}) + \rho(\vec{x}) \sum_{n=0}^{\infty} f_n(t) v_n(\vec{x}),$$

where

$$f_n(t) = \frac{1}{\|v_n(\vec{x})\|^2} \left\langle \frac{f(\vec{x}, t)}{\rho(\vec{x})} \middle| v_n(\vec{x}) \right\rangle_\rho. \tag{19.36}$$

Equating the Fourier coefficients on the left and on the right sides, we obtain the following equation for $\Theta_n(t)$:

$$\widehat{P}_t \Theta_n + \lambda_n \Theta_n = f_n(t), \qquad \Theta_n(0) = \Theta_n'(0) = 0. \tag{19.37}$$

The solution of Eq. (19.37) can be represented in the form

$$\Theta_n(t) = \int\limits_0^t \mathcal{K}_n(t, \tau) f_n(\tau) d\tau, \tag{19.38}$$

where $\mathcal{K}_n(t, \tau)$ is the Green's function of the Cauchy problem (19.37) (see Sec. "Fundamental solutions of linear operators" of Part II).

For the heat equation we obtain

$$\Theta_n(t) = e^{-a^2 \lambda_n t} \int\limits_0^t e^{a^2 \lambda_n \tau} f_n(\tau) d\tau \tag{19.39}$$

and

$$\mathcal{K}_n(t, \tau) = e^{-a^2 \lambda_n (t - \tau)}. \tag{19.40}$$

In the case of the wave equation, for the determination of the function $\Theta_n(t)$ we obtain an equation

$$\Theta_n'' + \lambda_n a^2 \Theta_n = f_n(t) \tag{19.41}$$

with the initial conditions (19.37).

We shall seek a solution of Eq. (19.41) by the Lagrange method:

$$\Theta_n(t) = p_n(t) \cos(a\sqrt{\lambda_n} t) + q_n(t) \sin(a\sqrt{\lambda_n} t).$$

Here, the functions $p_n(t)$ and $q_n(t)$ are determined by the system of equations

$$\begin{cases} p_n'(t)\cos(a\sqrt{\lambda_n}t) + q_n'(t)\sin(a\sqrt{\lambda_n}t) = 0, \\ p_n'(t)(-a\sqrt{\lambda_n})\sin(a\sqrt{\lambda_n}t) + q_n'(t)a\sqrt{\lambda_n}\cos(a\sqrt{\lambda_n}t) = f_n(t). \end{cases}$$

As a result, we obtain

$$p_n(t) = -\frac{1}{a\sqrt{\lambda_n}}\int\limits_0^t f_n(\tau)\sin(a\sqrt{\lambda_n}\tau)d\tau + p_n^0,$$

$$q_n(t) = \frac{1}{a\sqrt{\lambda_n}}\int\limits_0^t f_n(\tau)\cos(a\sqrt{\lambda_n}\tau)d\tau + q_n^0.$$

From the initial conditions (19.37) we find $q_n^0 = p_n^0 = 0$. Hence,

$$\Theta_n(t) = \frac{1}{a\sqrt{\lambda_n}}\int\limits_0^t f_n(\tau)\big[-\sin(a\sqrt{\lambda_n}\tau)\cos(a\sqrt{\lambda_n}t) +$$

$$+\cos(a\sqrt{\lambda_n}\tau)\sin(a\sqrt{\lambda_n}t)\big]d\tau = \frac{1}{a\sqrt{\lambda_n}}\int\limits_0^t f_n(\tau)\sin[a\sqrt{\lambda_n}(t-\tau)]d\tau.$$

Thus, for the wave equation we have

$$\mathcal{K}_n(t,\tau) = \frac{1}{a\sqrt{\lambda_n}}\sin[a\sqrt{\lambda_n}(t-\tau)]. \tag{19.42}$$

♦ A generalized function $\mathfrak{E}(\vec{x},\vec{y},t,\tau)$ is called the fundamental solution or Green's function of a mixed problem if this function, with fixed $\vec{y} \in \mathbb{R}^n$, $\tau \in \mathbb{R}$, satisfies the equation

$$\rho(\vec{x})\widehat{P}_t\mathfrak{E} = \widehat{L}_x\mathfrak{E} + \delta(\vec{x}-\vec{y})\delta(t-\tau) \tag{19.43}$$

and the homogeneous initial boundary condition

$$\Big[\alpha(\vec{x})\frac{\partial\mathfrak{E}}{\partial n} + \beta(\vec{x})\mathfrak{E}\Big]\Big|_{S_E} = 0, \quad \mathfrak{E}|_{t=0} = \mathfrak{E}_t|_{t=0} = 0, \tag{19.44}$$

$$\{\alpha^2(\vec{x}) + \beta^2(\vec{x})\}\big|_{S_E} \neq 0.$$

Statement 19.6. *The fundamental solution $\mathfrak{E}(\vec{x},\vec{y},t,\tau)$ of the mixed problem* (19.1) *can be represented in the form*

$$\mathfrak{E}(\vec{x},\vec{y},t,\tau) = \sum_{n=0}^{\infty}\mathcal{K}_n(t,\tau)v_n(\vec{x})v_n(\vec{y}). \tag{19.45}$$

Actually, putting in (19.36)

$$f(\vec{x},t) = \delta(\vec{x}-\vec{y})\delta(t-\tau),$$

we obtain (19.45) from (19.35).

Substituting the Fourier coefficients (19.36) into (19.35) and interchanging the summation and the integration, in view of the explicit form of the function $\mathfrak{E}(\vec{x},\vec{y},t,\tau)$ we arrive at the following statement:

Statement 19.7. *Let* $\mathfrak{E}(\vec{x},\vec{y},t,\tau)$ *be the fundamental solution of the mixed problem* (19.43) *and* (19.44). *Then the solution of the problem* (19.7) *can be represented in the form*

$$u_2(\vec{x},t) = \int\limits_0^t d\tau \int\limits_E \mathfrak{E}(\vec{x},\vec{y},t,\tau) f(\vec{y},\tau) d\vec{y}. \tag{19.46}$$

Statement 19.8. *The Green's functions* $\mathfrak{E}(\vec{x},\vec{y},t,\tau)$ (19.43) *and* $g(\vec{x},\vec{y},t)$ (19.25), (19.20) *are related as*

$$\mathfrak{E}(\vec{x},\vec{y},t,\tau) = \frac{1}{a_2(\tau)} g(\vec{x},\vec{y},t-\tau). \tag{19.47}$$

For the heat equation, the coefficient $a_2(\tau)$ *in relation* (19.47) *should be replaced by* $a_1(\tau)$.

Indeed, consider the function

$$u_2(\vec{x},t) = \int\limits_0^t d\tau \int\limits_E \frac{1}{a_2(\tau)} g(\vec{x},\vec{y},t-\tau) f(\vec{y},\tau) d\vec{y}, \tag{19.48}$$

where the function $g(\vec{x},\vec{y},t)$ is a solution of the problem (19.25). Then

$$\widehat{L}u_2(\vec{x},t) = \int\limits_0^t d\tau \int\limits_E \frac{1}{a_2(\tau)} \widehat{L}_x g(\vec{x},\vec{y},t-\tau) f(\vec{y},\tau) d\vec{y}.$$

Similarly,

$$\widehat{P}_t u_2(\vec{x},t) = \int\limits_0^t d\tau \int\limits_E \frac{1}{a_2(\tau)} \widehat{P}_t g(\vec{x},\vec{y},t-\tau) f(\vec{y},\tau) d\vec{y} +$$

$$+a_2(t) \int\limits_E \frac{1}{a_2(t)} g_t(\vec{x},\vec{y},0) f(\vec{y},t) d\vec{y} + a_2(t) \frac{\partial}{\partial t} \int\limits_E \frac{1}{a_2(t)} \hat{g}(\vec{x},\vec{y},0) f(\vec{y},t) d\vec{y} +$$

$$+a_1(t) \int\limits_E \frac{1}{a_2(t)} g(\vec{x},\vec{y},0) f(\vec{y},t) d\vec{y}.$$

In view of the definition (19.27), we obtain

$$\widehat{P}_t u_2(\vec{x},t) = \int\limits_0^t d\tau \int\limits_E \frac{1}{a_2(\tau)} \widehat{P}_t g(\vec{x},\vec{y},t-\tau) f(\vec{y},\tau) d\vec{y} + f(\vec{x},t).$$

Hence, by virtue of the arbitrariness of the function $f(\vec{y},\tau)$, the representation (19.46) is equivalent to the representation (19.48), and relation (19.47) is valid. For the heat equation, the proof is similar.

19.4 Nonhomogeneous boundary conditions

Finally, consider the problem (19.8). Let us show that it can be reduced to those considered above. Actually, let $\upsilon(\vec{x},t)$ be an arbitrary function satisfying the condition

$$\Big[\alpha(\vec{x})\frac{\partial\upsilon}{\partial n}+\beta(\vec{x})\upsilon\Big]\Big|_{S_E}=\mu(\vec{x},t)\big|_{S_E}. \tag{19.49}$$

We shall seek the solution of Eq. (19.8) in the form

$$u_3(\vec{x},t)=\upsilon(\vec{x},t)+w(\vec{x},t). \tag{19.50}$$

Then, to determine the function $w(\vec{x},t)$ we come to the following problem:

$$\begin{gathered}\rho(\vec{x})\widehat{P}_t w=\widehat{L}w+\tilde{f}(\vec{x},t), \qquad (19.51)\\ \Big(\alpha(\vec{x})\frac{\partial w}{\partial n}+\beta(\vec{x})w\Big)\Big|_{S_E}=0,\\ w\big|_{t=0}=-\upsilon\big|_{t=0},\quad w_t\big|_{t=0}=-\upsilon_t\big|_{t=0},\end{gathered}$$

where

$$\tilde{f}(\vec{x},t)=\widehat{L}\upsilon(\vec{x},t)-\rho(\vec{x})\widehat{P}_t\upsilon(\vec{x},t),$$

that is, for the function $w(\vec{x},t)$ we obtain the problems considered above. Then the function $w(\vec{x},t)$ can be written in the form

$$\begin{gathered}w(\vec{x},t)=\int\limits_0^t d\tau\int\limits_E\mathfrak{E}(\vec{x},\vec{y},t,\tau)\tilde{f}(\vec{y},\tau)d\vec{y}-\\ -\int\limits_E\mathfrak{g}(\vec{x},\vec{y},t)\upsilon(\vec{y},0)d\vec{y}-\int\limits_E g(\vec{x},\vec{y},t)\upsilon_t(\vec{y},0)d\vec{y}. \qquad (19.52)\end{gathered}$$

Correspondingly, for the heat equation, instead of (19.52), we obtain

$$\begin{gathered}w(\vec{x},t)=\int\limits_0^t d\tau\int\limits_E\mathfrak{E}(\vec{x},\vec{y},t,\tau)\tilde{f}(\vec{y},\tau)d\vec{y}-\int\limits_E g(\vec{x},\vec{y},t)\upsilon(\vec{y},0)d\vec{y}=\\ =\int\limits_0^t\frac{d\tau}{a_1(\tau)}\int\limits_E g(\vec{x},\vec{y},t-\tau)\tilde{f}(\vec{y},\tau)d\vec{y}-\int\limits_E g(\vec{x},\vec{y},t)\upsilon(\vec{y},0)d\vec{y}.\end{gathered}$$

For the operator $\widehat{P}_t=a^{-2}$, formula (19.52) can be written with the help of one Green's function $g(\vec{x},\vec{y},t)$

$$\begin{gathered}w(\vec{x},t)=\int\limits_0^t d\tau\int\limits_E g(\vec{x},\vec{y},t-\tau)\tilde{f}(\vec{y},\tau)d\vec{y}-\\ -\int\limits_E g_t(\vec{x},\vec{y},t)\upsilon(\vec{y},0)d\vec{y}-\int\limits_E g(\vec{x},\vec{y},t)\upsilon_t(\vec{y},0)d\vec{y}.\end{gathered}$$

20 The Sturm–Liouville problem and boundary value problems for stationary equations

Let us consider the following problem:

$$\widehat{L}u = f(\vec{x}), \tag{20.1}$$

$$\Big(\alpha(\vec{x})\frac{\partial u}{\partial n} + \beta(\vec{x})u\Big)\Big|_{S_E} = \varphi(\vec{x})\big|_{S_E}. \tag{20.2}$$

Seek a solution of the problem (20.1), (20.2) in the form

$$u(\vec{x}) = w(\vec{x}) + \upsilon(\vec{x}), \tag{20.3}$$

where $\upsilon(\vec{x})$ is an arbitrary function which is smooth together with its second derivatives and satisfies the condition

$$\Big[\alpha(\vec{x})\frac{\partial \upsilon}{\partial n} + \beta(\vec{x})\upsilon\Big]\Big|_{S_E} = \varphi(\vec{x})\big|_{S_E},$$

for instance, $\upsilon(\vec{x}) = \varphi(\vec{x})$, if the function $\varphi(\vec{x})$ satisfies the above conditions. Then we obtain for the function $w(\vec{x})$ an equation

$$\widehat{L}w = \bar{f}(\vec{x}) \tag{20.4}$$

with a homogeneous boundary condition

$$\Big[\alpha(\vec{x})\frac{\partial w}{\partial n} + \beta(\vec{x})w\Big]\Big|_{S_E} = 0. \tag{20.5}$$

Here,

$$\bar{f}(\vec{x}) = f(\vec{x}) - \widehat{L}\upsilon(\vec{x}).$$

We shall seek the solution of Eq. (20.4) in the form

$$w(\vec{x}) = \sum_{n=0}^{\infty} w_n v_n(\vec{x}), \tag{20.6}$$

where $v_n(\vec{x})$ are eigenfunctions of the operator $\widehat{L}$ in the region E. Substitution of (20.6) into Eq. (20.4) yields

$$\widehat{L}\sum_{n=0}^{\infty} w_n v_n(\vec{x}) = \bar{f}(\vec{x});$$

$$\sum_{n=0}^{\infty}(-\lambda_n)\rho(\vec{x})w_n v_n(\vec{x}) = \sum_{n=0}^{\infty}\rho(\vec{x})\alpha_n v_n(\vec{x}), \tag{20.7}$$

where

$$\alpha_n = -\frac{1}{\|v_n(x)\|^2}\Big\langle \frac{\bar{f}(\vec{x})}{\rho(\vec{x})}\Big| v_n(\vec{x})\Big\rangle_\rho, \tag{20.8}$$

and, equating the coefficients of identical functions $v_n(\vec{x})$, we obtain

$$\lambda_n w_n = -\alpha_n, \qquad n = \overline{0,\infty},$$

where λ_n are eigenvalues of the problem (18.1), (18.2). Assume that $\lambda_n \neq 0$, and, hence, the function $w(\vec{x})$ is

$$w(\vec{x}) = -\sum_{n=0}^{\infty} \frac{\alpha_n}{\lambda_n} v_n(\vec{x}) \tag{20.9}$$

and the solution of the original problem has the form

$$u(\vec{x}) = v(\vec{x}) - \sum_{n=0}^{\infty} \frac{\alpha_n}{\lambda_n} v_n(\vec{x}). \tag{20.10}$$

♦ A generalized function $g(\vec{x}, \vec{y})$ is called *the source function* or *Green's function of an internal boundary value problem* if this function, with fixed $\vec{y} \in \mathbb{R}^n$, satisfies the equation

$$\widehat{L}_x g(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y}) \tag{20.11}$$

and the homogeneous boundary condition

$$\Big\{\alpha(\vec{x}) \frac{\partial g(\vec{x}, \vec{y})}{\partial n_x} + \beta(\vec{x}) g(\vec{x}, \vec{y})\Big\}\Big|_{S_E} = 0, \tag{20.12}$$
$$\{\alpha^2(\vec{x}) + \beta^2(\vec{x})\}\big|_{S_E} \neq 0,$$

where $\vec{n}$ is a unit vector normal to the surface S and inner with respect to the region E (inner normal). For boundary conditions of the first ($\alpha(\vec{x})|_S \equiv 0$), second ($\beta(\vec{x})|_S \equiv 0$), and third kind, the function $g(\vec{x}, \vec{y})$ is called Green's function (source function) of the first, second, and third boundary value problem, respectively.

Statement 20.1. *The source function or Green's function $g(\vec{x}, \vec{y})$ of the internal boundary value problem* (20.11), (20.12) *can be represented in the form*

$$g(\vec{x}, \vec{y}) = \sum_{n=0}^{\infty} -\frac{\rho(\vec{y})}{\lambda_n \|v_n(\vec{x})\|^2} v_n(\vec{x}) v_n(\vec{y}). \tag{20.13}$$

Actually, substituting $f(\vec{x}) = \delta(\vec{x} - \vec{y})$ into (20.8), we obtain relation (20.13) from (20.9).

Statement 20.2. *Let $g(\vec{x}, \vec{y})$ be Green's function of the internal boundary value problem* (20.11), (20.12). *Then the solution of the homogeneous problem* (20.1), (20.2) $[\varphi(\vec{x}) = 0]$ *can be represented in the form*

$$u(\vec{x}) = \int_E g(\vec{x}, \vec{y}) \bar{f}(\vec{y}) d\vec{y}. \tag{20.14}$$

Actually, substituting the coefficients α_n (20.8) into (20.9) and interchanging the summation and the integration, in view of (20.13) and $\varphi(\vec{x}) = 0$, we make sure that the statement is true.

◊ From the explicit form of Green's function (20.13) it follows that this function is defined if the corresponding Sturm–Liouville problem has no trivial eigenvalue. Otherwise, generalized Green's functions can be introduced, similar to those arisen in the theory of ordinary differential equations (see Sec. "The boundary value problem for linear differential equations with a parameter" of Part II). For instance, for the second boundary condition $(\beta(\vec{x})\big|_{S_E} = 0)$

$$\big(\nabla, k(\vec{x})\nabla u\big) = f(\vec{x}), \qquad \frac{\partial u(\vec{x})}{\partial n}\Big|_{S_E} = 0 \tag{20.15}$$

there exists a generalized Green's function which satisfies, instead of (20.11), the condition

$$\Big\{\frac{\partial g(\vec{x},\vec{y})}{\partial n_x}\Big\}\Big|_{S_E} = -\frac{1}{S_0}, \tag{20.16}$$

where S_0 is the area of the surface S_E. Such Green's function is called a Neumann function (see also [17]).

CHAPTER 5

Elliptic Equations

21 Green's formulas

Theorem 21.1. *Let E be a region in the space $\mathbb{R}^3$ the boundary of which S_E is a closed piecewise smooth surface. Then, if functions $u(\vec{x})$ and $v(\vec{x})$ are continuous, together with their first and second derivatives, in the region E up to the boundary S_E, the following relations are valid:*

$$\int_E [u(\vec{x})\Delta v(\vec{x}) + (\nabla u(\vec{x}), \nabla v(\vec{x}))]d\vec{x} = \oint_{S_E} u(\vec{x})(\nabla v(\vec{x}), d\vec{S}); \tag{21.1}$$

$$\int_E [u(\vec{x})\Delta v(\vec{x}) - v(\vec{x})\Delta u(\vec{x})]d\vec{x} = \oint_{S_E} ([u(\vec{x})\nabla v(\vec{x}) - v(\vec{x})\nabla u(\vec{x})], d\vec{S}). \tag{21.2}$$

Relations (21.1) *and* (21.2) *are called, respectively, the first and the second Green's formulas.*

Proof. For the case of three variables, Ostrogradskii's formula is written as

$$\int_E \operatorname{div} \vec{A}(\vec{x})\, d\vec{x} = \oint_{S_E} (\vec{A}(\vec{x}),\, d\vec{S}). \tag{21.3}$$

On the right side, there is a surface integral of the second kind where $d\vec{S} = \vec{n}\, dS$ and the normal $\vec{n}$ is outer with respect to the closed surface S_E. Put in (21.3)

$$\vec{A}(\vec{x}) = u(\vec{x})\nabla v(\vec{x}),$$
$$\nabla v(\vec{x}) = \operatorname{grad} v(\vec{x}) = \Big(\frac{\partial v(\vec{x})}{\partial x_1}, \frac{\partial v(\vec{x})}{\partial x_2}, \frac{\partial v(\vec{x})}{\partial x_3}\Big),$$

where $u(\vec{x})$ and $v(\vec{x})$ are functions continuous together with their first derivatives in the region E. Then

$$\operatorname{div} \vec{A}(\vec{x}) = (\nabla, \vec{A}(\vec{x})) = (\nabla, (u(\vec{x}), \nabla v(\vec{x}))) = u(\vec{x})\Delta v(\vec{x}) + (\nabla u(\vec{x}), \nabla v(\vec{x})),$$

that is,

$$\operatorname{div} \vec{A}(\vec{x}) = u(\vec{x})\Delta v(\vec{x}) + (\nabla u(\vec{x}), \nabla v(\vec{x})).$$

Substituting this into (21.3), we obtain (21.1). Replace in (21.1) $u(\vec{x})$ by $v(\vec{x})$ and $v(\vec{x})$ by $u(\vec{x})$ to get

$$\int_E [v(\vec{x})\Delta u(\vec{x}) + (\nabla u(\vec{x}), \nabla v(\vec{x}))]d\vec{x} = \oint_{S_E} v(\vec{x})(\nabla u(\vec{x}), d\vec{S}). \tag{21.4}$$

Subtract from (21.1) the relation obtained to get (21.2). Thus, the theorem is proved.

◊ The requirements that are imposed by Theorem 21.1 on the functions $u(\vec{x})$ and $v(\vec{x})$ are determined by the type of the surface S_E for which Ostrogradskii's formula is valid. As shown below, these requirements can be weakened by strengthening the restrictions on the surface S_E.

◊ Green's formulas (21.3) and (21.1) remain valid for the case where the region E is bounded by several surfaces (i.e., it has a multiply connected boundary). In this case, the normal $\vec{n}$, which is outer to the region E, will be directed inward the surfaces that bound this region.

◊ Green's formulas (21.3) and (21.1) remain valid in the spaces $\mathbb{R}^n$, $n \geqslant 2$. In this case, by S_E is meant a closed plane curve for $n = 2$ or a closed hypersurface for $n > 3$.

22 Fundamental solutions of Helmholtz' and Laplace's equations

♦ An equation of the form

$$\Delta u - \lambda u = f(\vec{x}), \qquad \lambda = \text{const}, \quad \vec{x} \in \mathbb{R}^n \tag{22.1}$$

is called Helmholtz' equation.

♦ By the fundamental solution of or Green's function of Helmholtz' equation (22.1) is meant a generalized function $\mathcal{E}(\vec{x}, \vec{y})$ satisfying the equation

$$[\Delta_x - \lambda]\mathcal{E}(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y}), \qquad \vec{x}, \vec{y} \in \mathbb{R}^n, \tag{22.2}$$

where $\delta(\vec{x} - \vec{y}) = \delta(x_1 - y_1)\delta(x_2 - y_2) \cdots \delta(x_n - y_n)$ is the Dirac delta function, $\vec{x}$ is a variable, and $\vec{y}$ is a parameter. In the case $n = 3$ and positive λ, $\lambda = k_0^2$, Helmholtz' equation is used in nuclear physics (Yukawa's model). For negative λ, $\lambda = -k_0^2$, Helmholtz' equation arises in diffraction theory.

Theorem 22.1. *Green's function $\mathcal{E}(\vec{x}, \vec{y})$ enables us to find a particular solution of Eq. (22.1) for an arbitrary $f(\vec{x})$*

$$u(\vec{x}) = \int\limits_E \mathcal{E}(\vec{x}, \vec{y}) f(\vec{y}) d\vec{y}, \qquad E \subset \mathbb{R}^n. \tag{22.3}$$

Proof. Actually, multiply Eq. (22.2) by $f(\vec{y})$ and integrate the result with respect to $\vec{y}$ over the region E. Then,

$$[\Delta_x - \lambda] \int\limits_E \mathcal{E}(\vec{x}, \vec{y}) f(\vec{y}) d\vec{y} = \int\limits_E [\Delta_x - \lambda]\mathcal{E}(\vec{x}, \vec{y}) f(\vec{y}) d\vec{y} = \int\limits_E f(\vec{y})\delta(\vec{x} - \vec{y}) d\vec{y} = f(\vec{x}).$$

Thus, we arrive at (22.1), where $u(r)$ is determined from (22.3), which proves the theorem.

◊ Green's function is defined up to a summand $g_0(\vec{x}, \vec{y})$ which, with a fixed $\vec{y}$, is an arbitrary solution of the homogeneous equation

$$[\Delta_x - \lambda] g_0(\vec{x}, \vec{y}) = 0. \tag{22.4}$$

However, if we demand for Green's function to decrease at infinity, the solution of this problem for positive λ is unique. For the solution to be unique for negative λ, it is necessary to demand that Sommerfeld's conditions (16.18) be fulfilled.

◊ The function $f(\vec{x})$ has the sense of the source density of a given process (e.g., electrostatic charge density) in a given medium. Since the Dirac delta function $\delta(\vec{x} - \vec{y})$ is local and concentrated at the point $\vec{x} = \vec{y}$, then $\delta(\vec{x} - \vec{y}) = 0$ if $\vec{x} \neq \vec{y}$. Hence, the solution of Eq. (22.2) (Green's function) describes the influence of the point source located at the point $\vec{y}$.

Theorem 22.2. *The fundamental solution of Helmholtz' equation has the form*

$$\mathcal{E}_n(\vec{x},\vec{y}) = \begin{cases} -\dfrac{1}{4\pi}\dfrac{e^{\pm\sqrt{\lambda}|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} & \text{for} \quad n=3; \\ \pm\dfrac{i}{4}H_0^{(1,2)}[\sqrt{-\lambda}|\vec{x}-\vec{y}|] & \text{for} \quad n=2. \end{cases} \tag{22.5}$$

Proof. 1. Perform in Eq. (22.2) the change of variables $\vec{x}-\vec{y}=\vec{R}$. Then Eq. (22.2) takes the form

$$(\Delta_R-\lambda)\mathcal{E}_n(\vec{R},\vec{y})=\delta(\vec{R}).$$

Since the right side of the equation is independent of $\vec{y}$, its solution can be sought in the form

$$\mathcal{E}_n(\vec{R},\vec{y})=\mathcal{E}_n(\vec{R}),$$

and as a result we arrive at the equation

$$(\Delta_R-\lambda)\mathcal{E}_n(\vec{R})=\delta(\vec{R}). \tag{22.6}$$

Represent the function $\mathcal{E}_n(\vec{R})$ in terms of a Fourier integral:

$$\mathcal{E}_n(\vec{R})=\int\limits_{\mathbb{R}^n}\bar{\mathcal{E}}(\vec{p})e^{i(\vec{p},\vec{R})}d\vec{p}, \quad n=2,3.$$

Then

$$\Delta_R\mathcal{E}_n(\vec{R})=-\int\limits_{\mathbb{R}^n}p^2\bar{\mathcal{E}}_n(\vec{p})e^{i(\vec{p},\vec{R})}d\vec{p}, \qquad \vec{p}^{\,2}=p^2.$$

Substitution of this into (22.2) yields

$$-\int\limits_{\mathbb{R}^n}(\vec{p}^{\,2}+\lambda)\bar{\mathcal{E}}_n(\vec{p})e^{i(\vec{p},\vec{R})}d\vec{p}=\frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^n}e^{i(\vec{p},\vec{R})}d\vec{p}.$$

Here we have made use of the representation of the δ-function in terms of a Fourier integral (see Sec. "The Dirac delta function" of Part II)

$$\delta(\vec{R})=\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}e^{i(\vec{p},\vec{R})}d\vec{p}.$$

Hence,

$$\bar{\mathcal{E}}_n(\vec{p})=-\frac{1}{(2\pi)^n}\frac{1}{p^2+\lambda}. \tag{22.7}$$

After inverse Fourier transformation, we have

$$\mathcal{E}_n(\vec{R})=-\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}\frac{e^{i(\vec{p},\vec{R})}}{p^2+\lambda}d\vec{p}. \tag{22.8}$$

2. Calculate the integral (22.8) for $n=3$ and $\lambda=k_0^2$. To this end, pass to a spherical coordinate system, directing the vector p_3 along the vector $\vec{R}$. Then,

$(\vec{p},\vec{R}) = pR\cos\theta$, $\vec{p}^2 = p^2$, $d\vec{p} = p^2 dp\sin\theta\, d\theta\, d\varphi$, $p_1 = p\sin\theta\cos\theta$, $p_2 = p\sin\theta\sin\varphi$, $p_3 = p\cos\theta$, and

$$\mathcal{E}_3(\vec{R}) = -\frac{1}{(2\pi)^3}\int\limits_0^\infty p^2 dp\int\limits_0^\pi \sin\theta\, d\theta\int\limits_0^{2\pi} d\varphi\frac{e^{ipR\cos\theta}}{p^2+k_0^2} = \frac{1}{(2\pi)^2}\int\limits_0^\infty \frac{p^2 dp}{p^2+k_0^2}\int\limits_0^\pi e^{ipR\cos\theta} d(\cos\theta).$$

Calculate the integral

$$\int\limits_0^\pi e^{ipR\cos\theta} d(\cos\theta) = \frac{e^{ipR\cos\theta}}{ipR}\Big|_0^\pi = -\frac{2}{pR}\sin pR,$$

to get

$$\mathcal{E}_3(\vec{R}) = -\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{i(\vec{p},\vec{R})}}{p^2+k_0^2}d\vec{p} = -\frac{1}{2\pi^2 R}\int\limits_0^\infty \frac{p\sin pR}{p^2+k_0^2}dp =$$

$$= -\frac{1}{2\pi^2 R}\int\limits_0^\infty dx\frac{x\sin x}{x^2+(k_0R)^2} = -\frac{1}{4\pi R}e^{-k_0R}, \qquad (22.9)$$

in view of (see example 23.5 in Sec. "Applications of the theory of residues" of Part I)

$$\int\limits_0^\infty \frac{x\sin x}{x^2+a^2}dx = \frac{\pi}{2}e^{-a}, \qquad a > 0.$$

3. For $n = 2$, pass in the integral (22.8) to a polar coordinate system: $p_1 = p\cos\varphi$, $p_2 = p\sin\varphi$, $p = |\vec{p}|$, and $(\vec{p},\vec{R}) = \rho R\cos\varphi$. Then

$$\mathcal{E}_2(\vec{R}) = -\frac{1}{(2\pi)^2}\int\limits_0^\infty p\, dp\int\limits_0^{2\pi}\frac{e^{ipR\cos\varphi}}{p^2+k^2}d\varphi.$$

Using the representation of a Bessel function (III.8.7) with the help of Bessel's integral, we find

$$\mathcal{E}_2(\vec{R}) = -\frac{1}{2\pi}\int\limits_0^\infty \frac{pJ_0(Rp)\, dp}{p^2+k^2} = -\frac{1}{2\pi}K_0(Rk) = -\frac{i}{4}H_0^{(1)}(ikR). \qquad (22.10)$$

Here we have made use of formula (6.4) and relation (III.14). The other formulas of (22.5) are proved in a similar way or by the analytic extension method. For each specific case, the choice of the fundamental solution from (22.5) is conditioned by the statement of the problem.

♦ The function $\bar{\mathcal{E}}_n(\vec{p})$ (22.7) (Fourier transform of Green's function) is called the propagator of Eq. (22.2).

Example 22.1. Find the general solution of Eq. (22.4) depending only on $|\vec{x}|$.

Solution. Equation (22.4) for the function $g_0(\vec{x}) = g_0(|\vec{x}|)$, $|\vec{x}| = r$, has the form for $n = 3$ (in a spherical coordinate system)

$$\frac{1}{r^2}\frac{d}{dr}\Big(r^2\frac{dg_0}{dr}\Big) - \lambda g_0 = 0; \qquad (22.11)$$

for $n = 2$ (in a polar coordinate system)

$$\frac{1}{r}\frac{d}{dr}\Big(r\frac{dg_0}{dr}\Big) - \lambda g_0 = 0. \tag{22.12}$$

In view of the fact that

$$\frac{1}{r^2}\frac{d}{dr}\Big(r^2\frac{dg_0}{dr}\Big) = \frac{1}{r}\frac{d^2}{dr^2}(rg_0),$$

Eq. (22.11) can be written as

$$\frac{d^2}{dr^2}(rg_0) - \lambda(rg_0) = 0.$$

The general solution of this equation has the form

$$g_0(|\vec{x}|) = g_0(r) = \frac{1}{r}\big[C_1 e^{r\sqrt{\lambda}} + C_2 e^{-r\sqrt{\lambda}}\big]. \tag{22.13}$$

Equation (22.12), in turn, is Bessel's equation of order $\nu = 0$ with a parameter $-\lambda$:

$$r^2\frac{d^2g_0}{dr^2} + r\frac{dg_0}{dr} - \lambda r^2 g_0 = 0,$$

whose general solution has the form

$$g_0(|\vec{x}|) = g_0(r) = C_1 H_0^{(1)}(r\sqrt{-\lambda}) + C_2 H_0^{(2)}(r\sqrt{-\lambda}). \tag{22.14}$$

Since the homogeneous equation (22.4) everywhere except for the point $\vec{x} = \vec{y}$ coincides with (22.2), the obtained solutions (22.13) and (22.14) coincide, up to constant factors, with the corresponding fundamental solutions (22.5).

Corollary. The Green's function of Laplace's equation has the form

$$\mathcal{E}_n(\vec{x},\vec{y}) = \begin{cases} -\dfrac{1}{4\pi}\dfrac{1}{|\vec{x}-\vec{y}|}, & \vec{x},\vec{y} \in \mathbb{R}^3; \\ -\dfrac{1}{2\pi}\ln\dfrac{1}{|\vec{x}-\vec{y}|}, & \vec{x},\vec{y} \in \mathbb{R}^2. \end{cases} \tag{22.15}$$

Proof. Putting in relation (22.5) $\lambda = 0$ and taking into account the behavior of the Hankel functions $H_0^{(1,2)}(z)$ in the neighborhood of the point $z = 0$ (see Sec. "Asymptotic behavior of Bessel functions" of Part III)

$$H_0^{(1,2)}(z) \sim \pm\frac{2i}{\pi}\ln\frac{1}{|z|}, \quad z \to 0,$$

we obtain the Green's function of Laplace's equation (22.15).

◊ Relations (22.15) can be obtained from Laplace's equation with the help of a Fourier transform in the same way as relations (22.5).

◊ Obviously, the Green's function (22.15) satisfies the equation

$$\Delta_x\mathcal{E}_n(\vec{x},\vec{y}) = \delta(\vec{x}-\vec{y}), \qquad \Delta_x\frac{1}{|\vec{x}-\vec{y}|} = -4\pi\delta(\vec{x}-\vec{y}), \tag{22.16}$$

and a particular solution of Poisson's equation $\Delta u = f(\vec{x})$ has the form

$$u(\vec{x}) = -\frac{1}{4\pi}\int\frac{f(\vec{y})d\vec{y}}{|\vec{x}-\vec{y}|}, \quad \vec{x} \in \mathbb{R}^3. \tag{22.17}$$

◊ The Green's function of Laplace's equation (22.15) in electrostatics is the Coulomb potential of a point charge or a charged line and the Green's function of Helmholtz' equation (22.5) in nuclear physics is the Yukawa potential.

23 Harmonic functions

23.1 Harmonic functions and their properties

♦ A function $u(\vec{x})$ is called *harmonic* in a region $E \subset \mathbb{R}^n$ if it is continuous in E together with its partial derivatives up to the second ones inclusive and satisfies Laplace's equation $\Delta_n u = 0$ at all internal points of the region E.

◊ The definition of a harmonic function is given for an open region. If below we speak of a harmonic function in a closed region $\bar{E}$, this means that this function is harmonic in a broader open region G ($\bar{E} \subset G$).

◊ The above definition is valid for both finite and infinite regions. For unbounded regions, it is sometimes complemented with the condition

$$\lim_{|\vec{x}|\to\infty} |\vec{x}|^{n-2}u(\vec{x}) = \text{const} < \infty, \qquad n \geqslant 2. \tag{23.1}$$

As shown below, this requirement is fulfilled if a harmonic function is subject to the regularity conditions (17.7) or (17.8). We shall refer to functions of this type as harmonic functions regular at infinity.

◊ A regular harmonic function in an unbounded region $E \subset \mathbb{R}^n$ tends to zero as $|\vec{x}|$ tends to infinity along any curve belonging to E for $n > 2$ and is limited for $n = 2$.

◊ Note that for $n = 1$ harmonic functions satisfy the equation

$$\frac{d^2u}{dx^2} = 0$$

and are linear functions whose plots are straight lines. The theory of linear functions is considered in the course of calculus and here is of no interest. Nevertheless, the behavior of a straight line specified on the boundary of a region E, that is, at the points a and b of an interval $[a, b]$ well illustrates and, moreover, allows one to predict some properties of harmonic functions of dimension $n \geqslant 2$. Actually, two values of a linear function specified at two points (i.e., on a boundary) define uniquely, first, the explicit form of the function, i.e., the behavior of the function at all internal points of the interval, which corresponds to the theorem of integral representation, and, second, the mean value of the linear function on the interval $[a, b]$, which corresponds to the mean value theorem. Moreover, a linear function cannot reach either highest or lowest value at internal points of an interval $[a, b]$, which corresponds to the theorem of extremum or the maximum principle. Now we proceed to the consideration of these theorems, previously proving the following lemma:

Lemma 23.1. *If a function $u(\vec{x})$ is continuous together with its derivatives, up to the second ones inclusive, within a region E up to the boundary S_E, i.e., $u(\vec{x}) \in C^2(\bar{E})$, the following formulas are true:*

$$h(\vec{x}, E)u(\vec{x}) = \int\limits_E \mathcal{E}_n(\vec{x}, \vec{y})\Delta u(\vec{y})d\vec{y} \oint\limits_{S_E} \big\{\big[u(\vec{y})\nabla\mathcal{E}_n(\vec{x}, \vec{y}) - \mathcal{E}_n(\vec{x}, \vec{y})\nabla u(\vec{y})\big], d\vec{S}\big\}, \tag{23.2}$$

where

$$h(\vec{x}, E) = \begin{cases} 1 & \textit{for} \quad \vec{x} \in E, \\ 0 & \textit{for} \quad \vec{x} \notin E \end{cases}$$

is the characteristic function of a region E and

$$\mathcal{E}_n(\vec{x},\vec{x}) = \begin{cases} -\dfrac{1}{4\pi|\vec{x}-\vec{y}|} & \textit{for} \quad n=3, \\ -\dfrac{1}{2\pi}\ln\dfrac{1}{|\vec{x}-\vec{y}|} & \textit{for} \quad n=2. \end{cases}$$

♦ Relation (23.2) is called *the third Green's formula.*

Proof. For convenience, in the second Green's formula we put $\vec{x}=\vec{y}$:

$$\int\limits_E [u(\vec{y})\Delta_y v(\vec{y}) - v(\vec{y})\Delta_y u(\vec{y})]d\vec{y} = \oint\limits_{S_E} ([u(\vec{y})\nabla v(\vec{y}) - v(\vec{y})\nabla u(\vec{y})], d\vec{S}_y).$$

1. Choose $v(\vec{y}) = |\vec{x}-\vec{y}|^{-1}$ and take into account that $\Delta_y v(\vec{y}) = -4\pi\delta(\vec{x}-\vec{y})$. Then

$$-4\pi\int\limits_E u(\vec{y})\delta(\vec{x}-\vec{y})d\vec{y} = \int\limits_E \frac{1}{|\vec{x}-\vec{y}|}\Delta_y u(\vec{y})\, d\vec{y} + \oint\limits_{S_E}\Big(\Big[u(\vec{y})\nabla\frac{1}{|\vec{x}-\vec{y}|} - \frac{\nabla u(\vec{y})}{|\vec{x}-\vec{y}|}\Big], d\vec{S}_y\Big).$$

In view of

$$\int\limits_E u(\vec{y})\delta(\vec{x}-\vec{y})d\vec{y} = \begin{cases} u(\vec{x}), & \vec{x}\in E; \\ 0, & \vec{x}\notin E, \end{cases}$$

we obtain the statement of the lemma for $E\subset\mathbb{R}^3$.

2. Choose $v(\vec{y}) = \ln\frac{1}{|\vec{x}-\vec{y}|}$ and take into account that

$$\Delta_y \ln\frac{1}{|\vec{x}-\vec{y}|} = -2\pi\delta(\vec{x}-\vec{y}),$$

to get the statement of the lemma for $E\subset\mathbb{R}^2$.

Theorem 23.1 (of integral representation). *If a function $u(\vec{x})$ is harmonic within a region E, the following formula is true:*

$$u(\vec{x}) = -\oint\limits_{S_E}\big([u(\vec{y})\nabla\mathcal{E}_n(\vec{x},\vec{y}) - \mathcal{E}_n(\vec{x},\vec{y})\nabla u(\vec{y})], d\vec{S}_y\big); \qquad (23.3)$$

$$\vec{x}\in E\subset\mathbb{R}^n, \qquad n=2,3,$$

and the function $u(\vec{x})$ is infinitely differentiable at all internal points of the region E.

Proof follows immediately from Lemma 23.1 if in formula (23.2) one takes into account that $\Delta u = 0$. The infinite differentiability follows from the corresponding differentiability of the integral in the right side of relation (23.3) with respect to the parameter $\vec{x}$.

◊ Thus, a harmonic function in the region E is determined only by the values of $u(\vec{x})$ and $\nabla u(\vec{x})$ at the surface S_E.

Theorem 23.2 (of a normal derivative). *If a function $u(\vec{x})$ is harmonic within a region $E \subset \mathbb{R}^3$, then*

$$\oint_S (\nabla u(\vec{x}), d\vec{S}) = \oint_S \frac{\partial u(\vec{x})}{\partial n} dS = 0, \tag{23.4}$$

where S is an arbitrary closed surface completely situated in E. In other words, if a function $u(\vec{x})$ is harmonic within a region confined by a closed surface S, the flux $\nabla u(\vec{x})$ through this surface is equal to zero.

Proof. In the first Green's formula

$$\int_E [v\Delta u(\vec{x}) + (\nabla v(\vec{x}), \nabla u(\vec{x}))] d\vec{x} = \oint_S v(\vec{x})(\nabla u(\vec{x}), d\vec{S})$$

we put $v(\vec{x}) = 1$ and assume that the function $u(\vec{x})$ is harmonic. In view of $\Delta u(\vec{x}) = 0$, $v(\vec{x}) = 1$, and $\nabla v(\vec{x}) = 0$, we obtain (23.4). This proves the theorem.

Note that Theorem 23.2 holds true in the case where $u(\vec{x})$ is harmonic in E and continuous in $\bar{E}$. In this case, the surface S coincides with S_E.

Theorem 23.3 (of the mean). *For functions $u(\vec{x})$ which are harmonic within a sphere (circle) of radius R, the following relation is valid:*

$$u(\vec{x}) = \frac{1}{\pi(2R)^{n-1}} \oint_{|\vec{x}-\vec{y}|=R} u(\vec{y}) dS_y, \quad n = 2, 3. \tag{23.5}$$

Proof. For $n = 3$ we assume that S_E in (23.3) is a sphere of radius $R = |\vec{x} - \vec{y}|$. Then,

$$\oint_{|\vec{x}-\vec{y}|=R} \Big(\frac{\nabla u(\vec{y})}{|\vec{x}-\vec{y}|}, d\vec{S}_y\Big) = \frac{1}{R} \oint_{|\vec{x}-\vec{y}|=R} (\nabla u(\vec{y}), d\vec{S}_y) = 0.$$

The last integral is equal to zero by virtue of formula (23.4). Note that

$$\nabla \frac{1}{|\vec{x}-\vec{y}|} = -\frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|^3}\Big|_{|\vec{x}-\vec{y}|=R} = -\frac{\vec{R}}{R^3}.$$

For a sphere

$$d\vec{S}_y = \vec{n}\, dS_y = \frac{\vec{R}}{R} dS_y.$$

Hence, from (23.3) we obtain

$$u(\vec{x}) = \frac{1}{4\pi R^2} \oint_{|\vec{x}-\vec{y}|=R} u(\vec{y}) dS_y.$$

For $n = 2$, the proof is similar.

◊ The theorem of the mean states that the value of a harmonic function at a certain point $\vec{x}$ is equal to the mean value of this function on a sphere whose center is at the point $\vec{x}$ if the sphere is situated within the region of harmonicity.

Theorem 23.4 (of an extremum). *If a function $u(\vec{x})$ is harmonic within a closed region $\bar{E} = E + S_E$, where S_E is the boundary of E, this function cannot attain an extreme value at internal points of the region E, that is,*

$$\min_{\vec{x}\in S_E} u(\vec{x}) \leqslant u(\vec{x}) \leqslant \max_{\vec{x}\in S_E} u(\vec{x}), \quad \vec{x} \in E. \tag{23.6}$$

Relation (23.6) *is called the maximum principle.*

Proof. We perform the proof by the rule of contraries, assuming that $u(\vec{x})$ is harmonic in $\bar{E}$. Assume that the function $u(\vec{x})$ is not equal identically to a constant and a point $\vec{x}_0 \in E$ is the maximum point. Then we have

$$u_0 = u(\vec{x}_0) \geqslant u(\vec{x})$$

within some neighborhood of the point $\vec{x}_0$ belonging to the region of harmonicity. Take a small sphere of radius ε, which completely lies within this neighborhood. Then

$$u(\vec{x}_0) = \frac{1}{4\pi\varepsilon^2} \oint\limits_{|\vec{x}-\vec{x}_0|=\varepsilon} u(\vec{x})dS.$$

Since $u(\vec{x}) \leqslant u_0$, the following inequality should be fulfilled

$$u(\vec{x}_0) \leqslant \frac{u_0}{4\pi\varepsilon^2} \oint\limits_{|\vec{x}-\vec{x}_0|} dS \leqslant u_0.$$

Here we have used the relation

$$\oint\limits_{|\vec{x}-\vec{y}|=R} dS = 4\pi R^2.$$

Hence, only the equality is possible, and then $u(\vec{x}) = \text{const}$, which contradicts to the assumption and proves the statement. For the minimum point, the proof is similar.

Example 23.1. Show that if two functions, $u(\vec{x})$ and $v(\vec{x})$, harmonic in E and continuous in $\bar{E}$, satisfy, on the boundary S_E, the inequality

$$u(\vec{x}) \leqslant v(\vec{x}), \tag{23.7}$$

then this inequality holds true for all internal points of the region E.

Solution. Consider a function $w(\vec{x}) = u(\vec{x}) - v(\vec{x})$ harmonic in E, continuous in $\bar{E}$, and satisfying the condition $w(\vec{x})\big|_{S_E} \leqslant 0$. Suppose that at a certain internal point $\vec{x} \in E$, the inequality $w(\vec{x}) > 0$ takes place. This supposition, however, contradicts to the remark to Theorem 23.2, and, hence, inequality (23.7) is valid all over the closed region $\bar{E}$.

In conclusion, let us formulate some statements that follow from the remark to Theorem 23.2, which will be required to consider the boundary value problems for Laplace's and Poisson's equations.

Statement 23.1. *If $u(\vec{x})$ is harmonic in E and continuous in $\bar{E}$, then*

$$|u(x)| \leqslant \max_{\vec{x}\in S_E} |u(\vec{x})|, \quad \vec{x} \in \bar{E}. \tag{23.8}$$

Inequality (23.8) immediately follows from (23.6).

Statement 23.2. *If a harmonic function $u(\vec{x})$ is equal to zero on the boundary of a region E, then this function is identically equal to zero within this region.*

Statement 23.3. *If a sequence of functions $\{u_m(\vec{x})\}$, $m = \overline{0, \infty}$, harmonic in a region E and continuous in $\bar{E}$, converges uniformly on the boundary of the region E, then this sequence uniformly converges in $\bar{E}$ to some harmonic function.*

From inequality (23.8) it follows that

$$\lim_{\substack{k\to\infty\\ m\to\infty}} |u_k(\vec{x}) - u_m(\vec{x})| \leqslant \lim_{\substack{k\to\infty\\ m\to\infty}} \max_{\vec{x}\in S_E} |u_k(\vec{x}) - u_m(\vec{x})| = 0, \quad \vec{x} \in \bar{E},$$

which implies the first statement. The validity of the second one follows from the theorem of the mean: since the functions $u_m(\vec{x})$ satisfy equality (23.5), their limit satisfies the same equality, and thus the limiting function is harmonic.

Similar statements for unbounded regions will be formulated below after consideration of harmonic functions at infinity.

24 Separation of variables in Laplace's equation

This section is devoted to one of the most widely used methods for solving initial and boundary value problems – the Fourier method or, as it is referred to in accordance with the key idea, the method of separation of variables. This method appears to be efficient in those cases where, first, a partial differential equation in a chosen coordinate system admits separation of variables and, second, boundary conditions are specified on coordinate lines or surfaces of the given coordinate system. This enables one to select unique solutions from general solutions of ordinary differential equations in the corresponding variables and, as a consequence, to determine the unique solution of the original problem.

24.1 Separation of variables in Laplace's equation in Cartesian coordinates

Let us start with an example that clearly illustrates the heart of the method. In a rectangle $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, consider the following Dirichlet problem for a function $u(x, y)$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b; \tag{24.1}$$

$$u(x, 0) = f(x),\ u(x, b) = \varphi(x),\ u(0, y) = \psi(y),\ u(a, y) = \chi(y).$$

The functions $f(x)$, $\varphi(x)$, $\psi(y)$, and $\chi(y)$ are continuous on each side of the rectangle, while at its vertices, depending on the physical sense of the quantity $u(x, y)$, these functions may have discontinuities; otherwise the conditions of continuity

$$f(a) = \chi(0),\ \chi(b) = \varphi(a),\ \varphi(0) = \psi(b),\ \psi(0) = f(0) \tag{24.2}$$

should be fulfilled.

Let us consider in succession both cases.

I. The boundary conditions have discontinuities at the rectangle vertices.

In this case, one can immediately perform reduction of the problem.

♦ The procedure of transforming a problem to simpler ones is called reduction of the original problem.

Let us represent a function $u(x, y)$ as the sum of four harmonic functions

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y), \tag{24.3}$$

each taking given values on one of the sides and vanishing on the other three sides. As a result, we have four Dirichlet problems described by an equation

$$\Big[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big] u_i(x, y) = 0, \qquad i = \overline{1, 4} \tag{24.4}$$

with boundary conditions

$$\begin{aligned}
&u_1(x, 0) = f(x),\ u_1(x, b) = u_1(0, y) = u_1(a, y) = 0;\\
&u_2(x, 0) = 0,\ u_2(x, b) = \varphi(x),\ u_2(0, y) = u_2(a, y) = 0;\\
&u_3(x, 0) = u_3(x, b) = 0,\ u_3(0, y) = \psi(y),\ u_3(a, y) = 0;\\
&u_4(x, 0) = u_4(x, b) = u_4(0, y) = 0,\ u_4(a, y) = \chi(y).
\end{aligned} \tag{24.5}$$

Let us find one of the functions $u_i(x, y)$, for instance, $u_2(x, y)$. Following the idea of the method, we seek a particular solution of the problem in the form

$$u_2(x, y) = X(x)Y(y), \tag{24.6}$$

where $X(x)$ and $Y(y)$ are "separation" functions, each depending only on one variable. Substitution of (24.6) into (24.4) yields

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

The next step of the method of separation of variables is that an equation with already substituted "separation" functions is multiplied by a factor such that the resulting expression represents a sum of terms each depending on one variable. For the given case, such a factor is $[X(x)Y(y)]^{-1}$, and then we have

$$\frac{X''(x)}{(X(x)} + \frac{Y''(y)}{Y(y)} = 0.$$

The last equality is possible only in the case where each summand (group of summands) depending only on its variable is equal to a constant and the sum of these constants is equal to zero. Since we have only two variables, we should have two constants equal in magnitude and opposite in sign. Hence,

$$\frac{X''(x)}{(X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda, \quad \lambda = \text{const}\,.$$

As a result, we arrive at two ordinary differential equations

$$\begin{aligned}
X''(x) &= \lambda X(x),\\
Y''(y) &= -\lambda Y(y).
\end{aligned}$$

These equations should be complemented with boundary conditions which follow from the substitution of (24.6) into the second-line equalities of (24.5):

$$\begin{aligned}
X(x)Y(0) = 0, &\quad u_2(x, b) = \varphi(x),\\
X(0)Y(y) = 0, &\quad X(a)Y(y) = 0.
\end{aligned}$$

Since the multipliers $X(x)$ and $Y(y)$ with arbitrary x and y cannot vanish [otherwise $u_2(x, y) \equiv 0$], then

$$X(0) = X(a) = 0, \quad Y(0) = 0, \quad u_2(x, b) = \varphi(x). \tag{24.7}$$

Thus, we obtain for the functions $X(x)$ and $Y(y)$ the following problems:

$$X''(x) = \lambda X(x), \qquad X(0) = X(a) = 0; \tag{24.8}$$

$$Y''(y) = -\lambda Y(y), \qquad Y(0) = 0. \tag{24.9}$$

The problem (24.8) is a Sturm–Liouville problem for $X(x)$ whose solution is given in example III.2.2:

$$X_n(x) = a_n \sin\frac{\pi n x}{a}, \qquad \lambda_n = -\Big(\frac{\pi n}{a}\Big)^2, \quad n = \overline{1, \infty}. \tag{24.10}$$

The general solution of Eq. (24.9), in view of (24.10), has the form

$$Y_n(y) = b_n \sinh\frac{\pi n}{a}y + c_n \coth\frac{\pi n}{a}y.$$

If we take into account that $Y_n(0) = 0$, then we have

$$Y_n(y) = b_n \,\mathrm{sh}\,\frac{\pi n}{a}y$$

and, hence,

$$u_2^n(x, y) = X_n(x)Y_n(y) = a_n b_n \sin\frac{\pi n}{a}x \sinh\frac{\pi n}{a}y =$$
$$= \bar{a}_n \sin\frac{\pi n}{a}x \sinh\frac{\pi n}{a}y, \quad n = \overline{1, \infty}. \tag{24.11}$$

By virtue of the linearity of Laplace's equation, any sum of its solutions will also be a solution. Therefore, summing up all solutions of (24.11), we obtain the general form of the solution of Laplace's equation

$$u_2(x, y) = \sum_{n=1}^{\infty} \bar{a}_n \sin\frac{\pi n}{a}x \sinh\frac{\pi n}{a}y \tag{24.12}$$

that satisfies the homogeneous boundary conditions from the second line of (24.5).

The remaining undetermined arbitrary constants $\bar{a}_n = a_n b_n$ can be found using the last condition of (24.7):

$$u_2(x, y) = \sum_{n=1}^{\infty} \bar{a}_n \sin\frac{\pi n}{a}x \sinh\frac{\pi n b}{a} = \varphi(x). \tag{24.13}$$

Expand the function $\varphi(x)$ on the interval $]0, a[$ in a Fourier series of orthogonal functions (24.10)

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n \sin\frac{\pi n}{a}x; \quad \varphi_n = \frac{2}{a}\int\limits_0^a \varphi(x)\sin\frac{\pi n x}{a}dx. \tag{24.14}$$

Since an expansion in a Fourier series is unique, the equality (24.13) is valid only if the coefficients of the series (24.13) are equal to those of the series (24.14), that is,

$$\bar{a}_n = \frac{1}{\sinh(\pi n b/a)}\varphi_n, \tag{24.15}$$

and, hence,

$$u_2(x, y) = \sum_{n=1}^{\infty} \varphi_n \frac{\sinh(\pi n y/a)}{\sinh(\pi n b/a)} \sin\frac{\pi n}{a}x. \tag{24.16}$$

After similar calculations for the functions $u_1(x,y)$, $u_3(x,y)$, and $u_4(x,y)$, in view of (24.3), we finally obtain a solution of the problem

$$u(x,y)=\sum_{n=1}^{\infty}\Big\{\Big[\varphi_n \operatorname{sh}\frac{\pi n y}{a}+f_n \sinh\frac{\pi n(b-y)}{a}\Big]\frac{\sin(\pi n x/a)}{\operatorname{sh}(\pi n b/a)}+$$

$$+\Big[\chi_n \operatorname{sh}\frac{\pi n x}{b}+\psi_n \sinh\frac{\pi n(a-x)}{b}\Big]\frac{\sin(\pi n y/b)}{\sinh(\pi n a/b)}\Big\}, \qquad (24.17)$$

where f_n, χ_n, and ψ_n are the Fourier coefficients of the functions $f(x)$, $\chi(x)$, and $\psi(x)$, that are determined, analogous to (24.14), by the formulas

$$f_n(x)=\frac{2}{a}\int_0^a f(x)\sin\frac{\pi n x}{a}dx;$$

$$\chi_n=\frac{2}{b}\int_0^b \chi(y)\sin\frac{\pi n x}{b}dx; \qquad (24.18)$$

$$\psi_n=\frac{2}{b}\int_0^b \psi(y)\sin\frac{\pi n x}{b}dx.$$

II. The boundary conditions at the rectangle vertices are continuous.

As mentioned, in this case, the continuity conditions (24.2) should be fulfilled. Since the function $u(x,y)$ not necessarily vanishes at the rectangle vertices, the reduction of the problem directly for the function $u(x,y)$ proves to be impossible. Therefore, we shall seek a solution of the problem in the form

$$u(x,y)=u_0(x,y)+w(x,y), \qquad (24.19)$$

where $u_0(x,y)$ is a harmonic function which should be chosen so that the harmonic function $w(x,y)$ vanish at the rectangle vertices. This means that the function $u_0(x,y)$ should satisfy the following conditions:

$$\begin{array}{ll} u_0(0,0)=\psi(0)=f(0), & u_0(a,0)=f(a)=\chi(0),\\ u_0(a,b)=\varphi(a)=\chi(b), & u_0(0,b)=\varphi(0)=\psi(b), \end{array} \qquad (24.20)$$

Put

$$u_0(x,y)=A+Bx+Cy+Dxy. \qquad (24.21)$$

Such a function is harmonic with any real coefficients. We shall find these coefficients by subjecting relation (24.21) to conditions (24.20). As a result, we have

$$A=f(0),\quad B=\frac{f(a)-f(0)}{a},\quad C=\frac{\psi(b)-\psi(0)}{b},$$

$$D=\frac{[\varphi(a)-\varphi(0)]-[f(a)-f(0)]}{ab}. \qquad (24.22)$$

Thus, the change (24.19) leads to a Dirichlet problem for the function $w(x,y)$:

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0,\qquad 0<x<a,\quad 0<y<b, \qquad (24.23)$$

$$w(x,0)=\bar f(x),\quad w(x,b)=\bar\varphi(x),$$
$$w(0,y)=\bar\psi(y),\quad w(a,b)=\bar\chi(y).$$

The functions $\bar{f}(x)$, $\bar{\varphi}(x)$, $\bar{\psi}(y)$, and $\bar{\chi}(y)$ are given by the formulas

$$\begin{aligned} \bar{f}(x) = f(x) - u_0(x,0), \quad \bar{\varphi}(x) = \varphi(x) - u_0(x,b), \\ \bar{\psi}(y) = \psi(y) - u_0(0,y), \quad \bar{\chi}(y) = \chi(y) - u_0(a,y) \end{aligned} \tag{24.24}$$

and satisfy the continuity conditions

$$\bar{f}(a) = \bar{\chi}(0) = \bar{\chi}(b) = \bar{\varphi}(a) = \bar{\varphi}(0) = \bar{\psi}(b) = \bar{\psi}(0) = \bar{f}(0) = 0$$

which mean that the function $w(x,y)$, being continuous on the boundaries, vanishes at the rectangle vertices. For such a function, we can reduce the problem (24.23) to problems of the type (24.4) whose solutions are given by formulas (24.17) and (24.18) in which the functions $f(x)$, $\varphi(x)$, $\psi(y)$, and $\chi(y)$ should be replaced by the functions $\bar{f}(x)$, $\bar{\varphi}(x)$, $\bar{\psi}(y)$, and $\bar{\chi}(y)$.

◊ Note that the series (24.17) that determines the solution for the case of continuous boundary conditions is uniformly convergent in the rectangle, in contrast to similar series that determine the solution for discontinuous conditions which may lose their uniform convergence in the neighborhood of the vertices.

The Dirichlet problem for a rectangular parallelepiped is treated in a similar way.

Now we turn to the Neumann problem. To illustrate some peculiarities arising in this case, we consider the Neumann problem for a rectangle $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$:

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < a, \quad 0 < y < b; \tag{24.25}$$
$$u_x(0,y) = \frac{F}{b}, \quad u_y(x,0) = -\frac{F}{a}, \quad u_x(a,y) = u_y(x,b) = 0,$$

where $F = \text{const}$. Note at once that it is impossible to reduce this problem to two problems each having nonzero boundary conditions on three sides, since this would violate the necessary solvability condition for the Neumann problem

$$\oint\limits_{S_E} \frac{\partial u}{\partial n} dl = -\frac{F}{a}\int\limits_0^a dx + \frac{F}{b}\int\limits_0^b dy = 0. \tag{24.26}$$

Therefore, we seek a solution of the problem in the form

$$u(x,y) = u_0(x,y) + w(x,y), \tag{24.27}$$

where $u_0(x,y)$ is a harmonic polynomial

$$u_0(x,y) = A + Bx + Cy + Dxy + E(x^2 - y^2) \tag{24.28}$$

satisfying conditions (24.25), that is,

$$\begin{aligned} B + Dy = \frac{F}{b}, \quad B + Dy + 2Ea = 0, \\ C + Dx = -\frac{F}{a}, \quad C + Dx - 2Eb = 0, \end{aligned} \tag{24.29}$$

whence

$$u_0(x,y) = A + \frac{F}{2ab}[y^2 - 2by - (x^2 - 2ax)].$$

If the coefficient A remainrd undetermined is chosen in the form $A = F(b^2 - a^2)/(2ab)$, then

$$u_0(x,y) = \frac{F}{2ab}[(y-b)^2 - (x-a)^2]. \tag{24.30}$$

Substitution (24.27) reduces the problem (24.25) to a problem for the function $w(x, y)$

$$w_{xx} + w_{yy} = 0, \qquad 0 < x < a, \quad 0 < y < b;$$
$$w_x(0, y) = w_y(x, 0) = w_x(a, y) = w_y(x, b) = 0$$

whose solution is an arbitrary constant c. Thus, the solution of the Neumann problem has the form

$$u(x, y) = \frac{F}{2ab}[(y - b)^2 - (x - a)^2] + c. \tag{24.31}$$

Example 24.1. Find a stationary temperature distribution in a thin rectangular plate of length a and width b with a heat conductivity k if

(a) a given temperature is kept over the perimeter of the rectangle;

(b) heat in amount Q flows in through one side and flows out through the adjacent side.

Solution. (a) The mathematical formulation of the problem is similar to (24.1). Its solutions are given by formulas (24.17), (24.18) or (24.17)–(24.19).

(b) The mathematical formulation of the problem is similar to (24.25) where $F = Q/k$ should be put. The solution of the problem is given by formula (24.31).

24.2 Separation of variables in Laplace's equation in polar coordinates

Let us consider the first boundary value problem for a circle

$$\Delta_2 u = 0, \qquad u(x, y)\Big|_{\sqrt{x^2+y^2}=a} = f(x, y)\big|_{\sqrt{x^2+y^2}=a}, \tag{24.32}$$

where $\Delta_2 = \partial_x^2 + \partial_y^2$ and $f(x, y)$ is a given function. A problem for which $\sqrt{x^2 + y^2} < a$ is called internal and a problem for which $\sqrt{x^2 + y^2} > a$ is called external.

It is convenient to consider the problem (24.32) in polar coordinates

$$x = r\cos\varphi, \qquad y = r\sin\varphi.$$

In this case, the Laplace operator takes the form

$$\Delta_2 = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}. \tag{24.33}$$

(The proof is left for the student.)

Then the boundary condition (24.32) can be rewritten as

$$u(r, \varphi)|_{r=a} = f(\varphi), \tag{24.34}$$

where the function $f(\varphi)$ satisfies the Dirichlet conditions on the interval $[0, 2\pi]$. We shall solve Laplace's equation $\Delta_2 u = 0$ with the boundary conditions (24.34) by the method of separation of variables or the Fourier method. We seek a particular solution of the equation in the form

$$u(r, \varphi) = R(r)\Phi(\varphi), \tag{24.35}$$

where $R(r)$ and $\Phi(\varphi)$, "separation functions", depend only on one variable (r and φ, respectively). So we further omit the arguments of these functions. Substituting (24.35) into Laplace's equation in polar coordinates, we obtain

$$\frac{\Phi}{r}\frac{d}{dr}\Big(r\frac{dR}{dr}\Big) + \frac{R}{r^2}\Phi'' = 0.$$

Multiply the left and right sides of this equation by $r^2/(R\Phi)$, so that each term of the resulting expression depend only on one variable: r or φ. As a result, we obtain

$$\frac{r}{R}\frac{d}{dr}\Big(r\frac{dR}{dr}\Big) + \frac{\Phi''}{\Phi} = 0.$$

This equality is possible only if each summand is equal to a constant and the sum of these constants is equal to zero. Thus,

$$\frac{r}{R}\frac{d}{dr}\Big(r\frac{dR}{dr}\Big) = -\frac{\Phi''}{\Phi} = \lambda, \qquad \lambda = \mathrm{const}\,.$$

As a result, we arrive at the following equations for the functions $\Phi(\varphi)$ and $R(r)$:

$$\Phi'' + \lambda\Phi = 0, \qquad r\frac{d}{dr}(r\dot{R}) - \lambda R = 0. \tag{24.36}$$

Here the dot denotes the derivative with respect to the variable r. The function $u = \Phi R$ (24.35) is harmonic and therefore it reaches its highest value on the boundary of the region of harmonicity on condition that

$$|R(r)| < \infty, \qquad |\Phi(\varphi)| < \infty.$$

Moreover, the solution of Laplace's equation in polar coordinates should not change on addition of 2π to the angle:

$$\Phi(\varphi + 2\pi) = \Phi(\varphi). \tag{24.37}$$

For the determination of the function $\Phi(\varphi)$ and the parameter λ, we have arrived at the Sturm–Liouville problem (24.36), (24.37) whose solution is illustrated by example III.2.5. In this solution, $l = 2\pi$ should be put. Then

$$\Phi_n(\varphi) = A_n \cos n\varphi + B_n \sin n\varphi, \qquad \lambda = n^2. \tag{24.38}$$

We shall seek the function $R(r)$ in the form $R(r) = r^\mu$; then $\dot{R} = \mu r^{\mu-1}$ and $\ddot{R} = \mu(\mu-1)r^{\mu-2}$. Substituting this into (24.38), we get

$$r^2\ddot{R} + r\dot{R} - n^2 R = 0$$

and, hence, $\mu^2 = n^2$ or $\mu = \pm n$. Thus,

$$\begin{aligned} R_n(r) &= C_n r^n + B_n r^{-n}, \quad n \neq 0, \\ R_0(r) &= C_0 + B_0 \ln r, \qquad n = 0. \end{aligned} \tag{24.39}$$

Then from the condition $|R(r)| < \infty$ it follows that for the internal problem ($r < a$) $B_0 = B_n = 0$ and for the external problem ($r > a$) $B_0 = C_n = 0$. As a result, we arrive at two possible variants of the particular solution (24.35):

$$\begin{aligned} u_n^1(r,\varphi) &= r^n(A_n^1 \cos n\varphi + B_n^1 \sin n\varphi), \quad r \leqslant a; \\ u_n^2(r,\varphi) &= \frac{1}{r^n}(A_n^2 \cos n\varphi + B_n^2 \sin n\varphi), \quad r \geqslant a, \end{aligned} \tag{24.40}$$

where $n = \overline{0,\infty}$. By virtue of the linearity of the original equation, any sum of its solutions will also be a solution of this equation. Summing up all particular solutions (24.40), we obtain the most general form of Laplace's equation

$$u_1(r,\varphi) = \sum_{n=0}^{\infty} r^n (A_n^1 \cos n\varphi + B_n^1 \sin n\varphi), \quad r \leqslant a; \tag{24.41}$$

$$u_2(r,\varphi) = \sum_{n=0}^{\infty} r^{-n} (A_n^2 \cos n\varphi + B_n^2 \sin n\varphi), \quad r \geqslant a, \tag{24.42}$$

satisfying the conditions

$$|u(r,\varphi)| < \infty, \qquad u(r,\varphi + 2\pi) = u(r,\varphi). \tag{24.43}$$

The functions $u_1(r,\varphi)$ and $u_2(r,\varphi)$ are solutions of the internal and the external problem, respectively.

◊ It should also be noted that for a function harmonic within a ring $a < r < b$, the condition that the solution is finite is fulfilled and the general solution satisfying conditions (24.43) takes the form

$$u(r,\varphi) = a_0 + b_0 \ln r + \sum_{n=1}^{\infty}\Big[\Big(a_n r^n + \frac{b_n}{r^n}\Big)\cos n\varphi + \Big(c_n r^n + \frac{d_n}{r^n}\Big)\sin n\varphi\Big]. \tag{24.44}$$

Put in (24.41) $r = a$; then

$$u_1(a,\varphi) = \sum_{n=0}^{\infty} a^n (A_n^1 \cos n\varphi + B_n^1 \sin n\varphi) = f(\varphi). \tag{24.45}$$

Expand $f(\varphi)$ in a Fourier series, which is always possible since this function is periodic and satisfies the Dirichlet conditions, to get

$$f(\varphi) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos n\varphi + \beta_n \sin n\varphi,$$

where

$$\alpha_n = \frac{1}{\pi}\int_0^{2\pi} f(\psi)\cos n\psi\, d\psi, \quad \beta_n = \frac{1}{\pi}\int_0^{2\pi} f(\psi)\sin n\psi\, d\psi. \tag{24.46}$$

Since an expansion in a Fourier series is unique, the equality (24.45) is possible only if

$$A_n^1 = \frac{\alpha_n}{a^n}, \quad B_n^1 = \frac{\beta_n}{a^n}.$$

Perform similar manipulations for $u_2(r,\varphi)$. Finally, we have

$$u_{1,2}(r,\varphi) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}\Big(\frac{r}{a}\Big)^{\pm n}(\alpha_n \cos n\varphi + \beta_n \sin n\varphi). \tag{24.47}$$

◊ If α_n and β_n are Fourier coefficients of finite and continuous functions, then the series for $u_1(r,\varphi)$ with $r < a$ (for u_2 with $r > a$) converges uniformly.

Example 24.2. Find a function harmonic within a circle of radius a, such that (1) $u|_{r=a} = \varphi$ and (2) $u|_{r=a} = A\sin^3\varphi + B$, where A and B are some constants.

Solution. (1) The mathematical formulation of the problem is

$$\Delta u = 0, \qquad u|_{r=a} = \varphi, \quad r < a, \quad 0 \leqslant \varphi \leqslant 2\pi.$$

The solution of this problem is given by the function $u_1(r,\varphi)$ (24.47). Let us find the coefficients of the Fourier series for the function $f(\varphi) = \varphi$. From (24.46) we have

$$\alpha_n = \frac{1}{\pi}\int\limits_0^{2\pi} \varphi\cos n\varphi\, d\varphi.$$

Integrate this by parts, putting $U = \varphi$, $dU = d\varphi$, $dV = \cos n\varphi\, d\varphi$, and $V = (\sin n\varphi)/n$, to get

$$\alpha_n = \frac{1}{\pi}\Big[n\varphi\sin n\varphi\Big|_0^{2\pi} - \int\limits_0^{2\pi}\frac{\sin n\varphi}{n}d\varphi\Big] = 0,$$

where $n = \overline{1,\infty}$. Similarly,

$$\beta_n = \frac{1}{\pi}\int\limits_0^{2\pi}\varphi\sin n\varphi\, d\varphi = -\frac{2}{n}, \quad \alpha_0 = \frac{1}{\pi}\int\limits_0^{2\pi}\varphi\, d\varphi = 2\pi.$$

Finally, we obtain

$$u(r,\varphi) = \pi - 2\sum_{n=1}^{\infty}\Big(\frac{r}{a}\Big)^n\frac{\sin n\varphi}{n} = 2\,\mathrm{arctg}\,\frac{a - r\cos\varphi}{r\sin\varphi}.$$

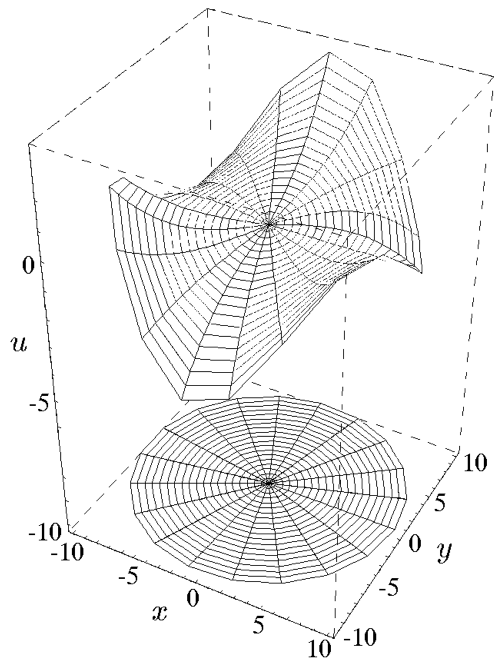

Fig. 31.

(2) The mathematical formulation of the problem is

$$\Delta u = 0, \qquad u|_{r=a} = A\sin^3\varphi + B,$$
$$r < a, \quad 0 \leqslant \varphi \leqslant 2\pi,$$

and its solution is given by formula (24.47), where the exponent n should be taken positive. In view of the fact that

$$\sin^3\varphi = \frac{3}{4}\sin\varphi - \frac{1}{4}\sin 3\varphi,$$

mere comparison yields

$$\alpha_0 = 2B, \qquad \alpha_k = 0, \quad k = \overline{1,\infty};$$
$$\beta_1 = \frac{3A}{4}, \quad \beta_2 = 0, \quad \beta_3 = -\frac{A}{4},$$
$$\beta_k = 0, \quad k = \overline{4,\infty}.$$

The same result can be obtained from (24.46) in view of the orthogonality relationship for trigonometric functions. Thus,

$$u(r,\varphi) = B + \frac{3A}{4}\frac{r}{a}\sin\varphi - \frac{A}{4}\Big(\frac{r}{a}\Big)^3\sin 3\varphi.$$

The plot of this function for $A = 1$, $B = 0$, and $a = 10$ is given in Fig. 31.

Example 24.3. Find a stationary temperature distribution in an infinite cylinder whose generatrix is parallel to the Oz axis and the directrix is the boundary of a circular sector of radius b of angle α, $0 < \alpha < 2\pi$, lying in the plane $z = 0$ if its boundary is kept at a temperature $u(r, 0) = u(r, \alpha) = 0$, $u(b, \varphi) = \varphi$.

Solution. The mathematical formulation of the problem is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0, \qquad (24.48)$$
$$u(r, 0) = u(r, \alpha) = 0, \qquad u(b, \varphi) = \varphi.$$

Since the boundary conditions are independent of z, then $u(r, \varphi, z) = u(r, \varphi)$. We seek a solution of the problem by the method of separation of variables:

$$u(r, \varphi) = R(r)\Phi(\varphi). \qquad (24.49)$$

Substituting this into (24.48) and multiplying the resulting equation by $r^2/(R\Phi)$, we obtain

$$r^2 \frac{R'' + R'/r}{R} + \frac{\ddot{\Phi}}{\Phi} = 0.$$

Upon separation of variables, we have for the function $\Phi(\varphi)$ the following Sturm–Liouville problem:

$$\ddot{\Phi} = \lambda\Phi, \qquad \Phi(0) = \Phi(\alpha) = 0. \qquad (24.50)$$

Its solution is given in example III.2.2 and has the form

$$\Phi_n = A_n \sin\frac{\pi n \varphi}{\alpha}, \qquad n = \overline{1, \infty}, \quad \lambda_n = -\Big(\frac{\pi n}{\alpha}\Big)^2. \qquad (24.51)$$

Then for the determination of the function $R(r)$ we obtain the equation

$$r^2 R'' + rR' - \Big(\frac{\pi n}{\alpha}\Big)^2 R = 0, \qquad |R(r)| < \infty, \qquad (24.52)$$

whose solution is sought in the form

$$R(r) = r^\mu.$$

Substitution of this into (24.52) yields

$$\mu(\mu - 1) + \mu - \Big(\frac{\pi n}{\alpha}\Big)^2 = 0,$$

whence

$$\mu_{1,2} = \pm\frac{\pi n}{\alpha},$$

and the general solution of Eq. (24.52) can be represented in the form

$$R_n(r) = B_n r^{\pi n/\alpha} + C_n r^{-\pi n/\alpha}.$$

From the condition $|R(r)| < \infty$ for $r \in [0, b]$ we find that $C_n \equiv 0$. Then

$$R_n(r) = B_n r^{\pi n/\alpha}. \qquad (24.53)$$

Substitution of (24.51) and (24.53) into (24.49) and summation over n yields

$$u(r,\varphi)=\sum_{n=1}^{\infty}\bar{A}_n r^{\pi n/\alpha}\sin\frac{\pi n\varphi}{\alpha}, \tag{24.54}$$

where $\bar{A}=A_nB_n$. Substitute (24.54) into the boundary condition (24.48)

$$u(b,\varphi)=\sum_{n=1}^{\infty}\bar{A}_n b^{\pi n/\alpha}\sin\frac{\pi n\varphi}{\alpha}=\varphi$$

and expand the right side in a Fourier series in orthogonal functions (24.51). Equating coefficients of identical functions, we find

$$\bar{A}_n b^{\pi n/\alpha}=\frac{\int\limits_0^\alpha \varphi\sin(\pi n\varphi/\alpha)d\varphi}{\int\limits_0^\alpha \sin^2(\pi n\varphi/\alpha)du}=(-1)^{n+1}\frac{2\alpha}{\pi n}.$$

Finally, we obtain the temperature distribution

$$u(r,\varphi)=\frac{2\alpha}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\Big(\frac{r}{b}\Big)^{\pi n/\alpha}\sin\frac{\pi n\varphi}{\alpha}.$$

Let us now consider the Neumann problem

$$\Delta_2 u(r,\varphi)=0,\qquad E: r<R; \tag{24.55}$$
$$\frac{\partial u}{\partial n}\Big|_{r=a}=f(\varphi),$$

where n is a normal to a circle of radius R, which is outer with respect to the region E. If the function $f(\varphi)$ satisfies the solvability condition (24.26), we seek a solution of the problem in the form

$$u_{1,2}(r,\varphi)=\sum_{k=0}^{\infty}r^{\pm k}(\bar{A}_k\cos k\varphi+\bar{B}_k\sin k\varphi),$$

where arbitrary constants $\bar{A}_k$ and $\bar{B}_k$ should be determined from the boundary conditions (24.55). Since the direction of the normal $\vec{n}$ coincides with that of the radius vector $\vec{r}$, then

$$\frac{\partial u}{\partial n}=\frac{\partial u}{\partial r}.$$

Since this condition does not determine the coefficient $\bar{A}_0$, then, up to an arbitrary constant $C=\bar{A}_0$, we find

$$u_1(r,\varphi)=\sum_{k=1}^{\infty}\frac{r^k}{ka^{k-1}}(\alpha_k\cos k\varphi+\beta_k\sin k\varphi)+C \tag{24.56}$$

for the internal problem and

$$u_2(r,\varphi) = \sum_{k=1}^{\infty} \frac{a^{k-1}}{kr^k}(\alpha_k \cos k\varphi + \beta_k \sin k\varphi) + C \tag{24.57}$$

for the external problem. The coefficients α_k and β_k are determined, as before, by formulas (24.46).

The third boundary value problem is solved similarly.

Note that the Kelvin transform with respect to a circle of radius a leaves the circle invariable and, moreover, being conformal, conserves the angles between the circle and a normal. Therefore, we can derive solutions of external problems from solutions of internal problems with the use of the Kelvin transform, i.e., by the change $r \to a^2/r$. One can readily check that such a change in (24.32) transforms $u_1(r,\varphi)$ into $u_2(r,\varphi)$ and (24.56) into (24.57) and *vice versa*.

Example 24.4. For the Neumann problem (24.55), select incorrectly formulated problems: (a) $f(\varphi) = A = \text{const}$; (b) $f(\varphi) = \cos\varphi$, and (c) $f(\varphi) = \sin^2\varphi$.

Solution. For a Neumann problem, the solvability condition (24.26) should be fulfilled. In polar coordinates, this condition for a circle $r = a$ takes the form

$$I = \int\limits_0^{2\pi} f(\varphi)d\varphi = 0, \tag{24.58}$$

that is,

$$\text{(a)} \quad I = 2\pi A, \quad \text{(b)} \quad I = 0, \quad \text{(c)} \quad I = \pi.$$

Hence, problems (a) and (c) are incorrectly formulated problems. This means that there are no harmonic functions satisfying conditions (a) and (c) on a circle.

24.3 Separation of variables in Laplace's equation in cylindrical coordinates

♢ For an unbounded cylinder and with boundary conditions are independent of the variable z, the solution of the problem has the form

$$u(r,\varphi,z) = u(r,\varphi).$$

For the function $u(r,\varphi)$, we have a two-dimensional problem whose solution has been obtained above (see, e.g., example 24.3).

Let us consider several problems of finding harmonic functions within a cylinder of radius a and finite height h, which can be solved by the method of separation of variables.

The general statement of such a problem has the form

$$\begin{gathered} \Delta u = 0, \qquad 0 \leqslant r < a, \quad 0 < z < h, \quad 0 \leqslant \varphi < 2\pi; \\ u(r,\varphi+2\pi,z) = u(r,\varphi,z), \qquad u(a,\varphi,z) = f_1(\varphi,z), \\ u(r,\varphi,0) = f_2(r,\varphi), \qquad u(r,\varphi,h) = f_3(r,\varphi) \end{gathered}$$

and admits the following reduction for $u = u_1 + u_2$:

$$\begin{gathered} \Delta u_1 = 0, \quad u_1(a,\varphi,z) = f_1(\varphi,z), \\ u_1(r,\varphi,0) = u_1(r,\varphi,h) = 0 \end{gathered}$$

and

$$\Delta u_2 = 0, \quad u_2(a, \varphi, z) = 0,$$
$$u_2(r, \varphi, 0) = f_1(r, \varphi), \qquad u_2(r, \varphi, h) = f_2(r, \varphi).$$

The question of taking into account the dependence on the azimuthal angle φ was considered in the preceding section; so we turn at once to problems with boundary conditions independent of the variable φ.

Example 24.5. Let there be given a cylinder of radius $r = a$ and height h with the lateral surface and the bottom kept at zero temperature and the top kept at temperature T. Find a stationary temperature distribution in the cylinder.

Solution. The mathematical formulation of the problem is

$$\Delta u = 0, \qquad 0 < z < h, \quad r < a, \quad 0 \leqslant \varphi < 2\pi;$$
$$u|_{z=0} = 0, \quad u|_{r=a} = 0, \quad u|_{z=h} = T.$$

Write down the Laplace operator in cylindrical coordinates:

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2}.$$

Since the boundary conditions are independent of the variable φ, the solution of the problem can be sought in the form $u(r, \varphi, z) = u(r, z)$, that is,

$$\frac{\partial u}{\partial \varphi} = 0,$$

and, hence, Laplace's equation takes the form

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0.$$

We seek a solution of the problem by the Fourier method, putting

$$u(r, z) = R(r)Z(z).$$

Denote

$$R' = \frac{dR}{dr}, \qquad \dot{Z} = \frac{dZ}{dz}.$$

Then

$$\frac{1}{r}\frac{\partial}{\partial r} r R' Z + R\ddot{Z} = 0.$$

Separation of variables yields

$$\frac{R''}{R} + \frac{R'}{rR} = -\frac{\ddot{Z}}{Z} = \lambda, \qquad \lambda = \text{const}$$

or

$$R'' + \frac{R'}{r} = R\lambda; \qquad \ddot{Z} + \lambda Z = 0.$$

From the boundary condition

$$u(a, z) = R(a)Z(z) = 0$$

we find $R(a) = 0$. Since the function $u(r, z)$ has the sense of temperature, then

$$|R(r)| < \infty \quad \text{for} \quad r \leqslant a.$$

As a result, the function $R(r)$ is a solution of the Sturm–Liouville problem for Bessel's equation

$$R'' + \frac{1}{r}R' - \lambda R = 0, \quad \lim_{r\to+0} |R(r)| < \infty, \quad R(a) = 0.$$

For the eigenvalues and eigenfunctions, we have, respectively,

$$\lambda = -\Big(\frac{\alpha_n^0}{a}\Big)^2, \qquad n = \overline{1, \infty}$$

and

$$R_n(r) = A_n J_0\Big(\alpha_n^0 \frac{r}{a}\Big),$$

where α_n^0 are roots of the function $J_0(x)$ (see Sec. "The Sturm–Liouville problem for the Bessel equation" of Part III). As a result, the equation for the function $Z(z)$ takes the form

$$\ddot{Z} - (\alpha_n^0)^2 Z = 0, \qquad n = \overline{1, \infty}.$$

Hence,

$$Z_n(z) = B_n \operatorname{sh} \alpha_n^0 z + C_n \operatorname{ch} \alpha_n^0 z$$

and

$$u_n(r, z) = A_n J_0\Big(\alpha_n^0 \frac{r}{a}\Big)[B_n \operatorname{sh} \alpha_n^0 z + C_n \operatorname{ch} \alpha_n^0 z].$$

Summing up over n gives us

$$u(r, z) = \sum_{n=1}^{\infty} J_0\Big(\alpha_n^0 \frac{r}{a}\Big)[\bar{B}_n \operatorname{sh} \alpha_n^0 z + \bar{C}_n \operatorname{ch} \alpha_n^0 z],$$

where

$$\bar{B}_n = A_n B_n, \qquad \bar{C}_n = A_n C_n.$$

From the boundary conditions we obtain

$$u|_{z=0} = \sum_{n=0}^{\infty} \bar{C}_n J_0\Big(\alpha_n^0 \frac{r}{a}\Big) = 0.$$

Hence, $\bar{C}_n = 0$. Analogously,

$$u|_{z=h} = \sum_{n=1}^{\infty} \bar{C}_n J_0\Big(\alpha_n^0 \frac{r}{a}\Big) \operatorname{sh}(\alpha_n^0 h) = T.$$

Expand the function $f(r)$ in a Fourier–Bessel series on the interval $]0, a[$:

$$f(r) = T = \sum_{n=1}^{\infty} \beta_n J_0\Big(\alpha_n^0 \frac{r}{a}\Big),$$

where

$$\beta_n = \frac{1}{a^2} \frac{2}{[J_0'(\alpha_n^0)]^2} \int_0^a r f(r) J_0\Big(\alpha_n^0 \frac{r}{a}\Big) dr.$$

Then

$$\bar{C}_n = -\frac{\beta_n}{2\,\mathrm{sh}\,\alpha_n^0 h}.$$

Thus,

$$u(r,z) = \sum_{n=1}^{\infty} J_0\Big(\alpha_n^0 \frac{r}{a}\Big)\beta_n \frac{\mathrm{sh}\,\alpha_n^0 z}{\mathrm{sh}\,\alpha_n^0 h}.$$

To determine the coefficient β_n of the Fourier–Bessel series of the function $f(r) = T$, calculate the integral

$$I_n = \frac{1}{a^2}\int\limits_0^a rTJ_0\Big(\alpha_n^0 \frac{r}{a}\Big)dr.$$

Perform in this integral the change of variables $\alpha_n^0 r/a = t$. Then $dr = (a/\alpha_n^0)dt$. For the new limits of integration, we obtain $r_1 = 0$, $t_1 = 0$, $r_2 = a$, and $t_2 = \alpha_n^0$ and then

$$I_n = \frac{T}{(\alpha_n^0)^2}\int\limits_0^{\alpha_n^0} tJ_0(t)dt = \frac{T}{(\alpha_n^0)^2}\int\limits_0^{\alpha_n^0}[tJ_1(t)]'dt =$$
$$= \frac{T}{(\alpha_n^0)^2}[tJ_1(t)]\Big|_0^{\alpha_n^0} = \frac{T}{(\alpha_n^0)^2}[\lambda_n^0 J_1(\alpha_n^0)] = -\frac{TJ_0'(\alpha_n^0)}{\alpha_n^0}.$$

Finally, we have

$$\beta_n = \frac{I_n}{[J_0'(\alpha_n^0)]^2} = -\frac{2T}{(\alpha_n^0)J_0'(\alpha_n^0)}$$

and

$$u(r,z) = \sum_{n=1}^{\infty} -J_0\Big(\alpha_n^0\frac{r}{a}\Big)\frac{2T}{\alpha_n^0 J_0'(\alpha_n^0)}\frac{\mathrm{sh}\,\alpha_n^0 z}{\mathrm{sh}\,\alpha_n^0 h} = \sum_{n=1}^{\infty}\frac{2T}{\alpha_n^0}\frac{\mathrm{sh}\,\alpha_n^0 z}{\mathrm{sh}\,\alpha_n^0 h}\frac{J_0(\alpha_n^0 r/a)}{J_1(\alpha_n^0)}.$$

Example 24.6. Let there be given a cylinder of radius $r = a$ and height h with the lateral surface kept at temperature $\sin \pi m z$ $(m = \overline{1,\infty})$ and both the top and bottom kept at zero temperature. Find the stationary temperature distribution in side of the cylinder.

Solution. The mathematical formulation of the problem is

$$\Delta u = 0, \qquad 0 < z < h, \quad r < a,$$
$$u|_{z=0} = u|_{z=h} = 0, \quad u|_{r=a} = \sin \pi m z.$$

Seek a solution of the problem by the Fourier method, putting

$$u(r,z) = R(r)Z(z).$$

As in the preceding example, we have for the functions $R(r)$ and $Z(z)$ the following problems:

$$R'' + \frac{R'}{r} = \lambda R; \qquad |R(r)| < \infty \quad r \leqslant a, \tag{24.59}$$

$$\ddot{Z} + \lambda Z = 0, \qquad z(0) = z(h) = 0. \tag{24.60}$$

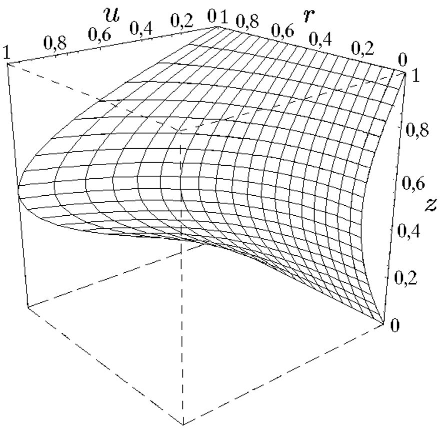

Fig. 32.

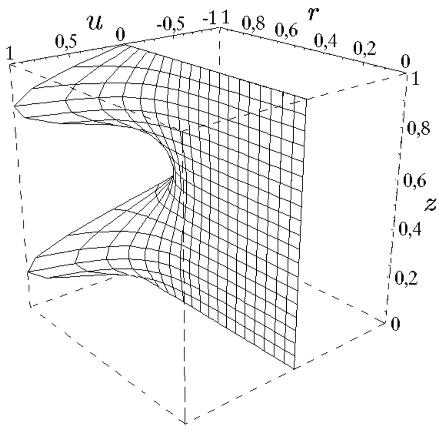

Fig. 33.

The eigenvalues and eigenfunctions of the Sturm–Liouville problem (24.60) obtained earlier (see example III.2.2) have the form

$$\lambda = (\frac{\pi n}{h})^2, \quad Z_n(z) = A_n \sin\frac{\pi n z}{h}, \qquad n = \overline{1, \infty}.$$

Then the general solution of Eq. (24.59) is

$$R_n(r) = B_n I_0\Big(\frac{\pi n r}{h}\Big) + C_n K_0\Big(\frac{\pi n r}{h}\Big),$$

and the boundedness condition (24.59) gives us $C_n = 0$. Hence,

$$u(r, z) = \sum_{n=1}^{\infty} \bar{A}_n I_0\Big(\frac{\pi n r}{h}\Big)\sin\frac{\pi n z}{h},$$

where $\bar{A}_n = A_n B_n$. From the boundary conditions we get

$$u|_{r=a} = \sum_{n=1}^{\infty} \bar{A}_n I_0\Big(\frac{\pi n a}{h}\Big)\sin\frac{\pi n z}{h} = \sin \pi m z.$$

Thus,

$$\bar{A}_n I_0\Big(\frac{\pi n a}{h}\Big) = \delta_{nm},$$

and for the solution of the original problem we obtain

$$u(r, z) = \frac{I_0(\pi m r/h)}{I_0(\pi m a/h)}\sin\frac{\pi m z}{h}.$$

The plot of this function for $a = 1$, $h = 1$, and $m = 1$ is given in Fig. 32 and that for $a = 1$, $h = 1$, and $m = 3$ in Fig. 33.

24.4 Separation of variables in Laplace's equation in spherical coordinates

Consider the problem of finding a harmonic function by applying the method of separation of variables in spherical coordinates.

Example 24.7. Find a function satisfying the condition $u_r|_{r=1} = \sin(\pi/4 - \varphi)\sin\theta$ on a sphere of unit radius and being harmonic (a) outside the sphere or (b) inside the sphere.

Solution. Write down Laplace's equation in spherical coordinates. Then the problem of the determination of the function $u(r, \theta, \varphi)$ is formulated as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial u}{\partial\theta}\Big) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} = 0; \tag{24.61}$$

$$u_r(r, \theta, \varphi)\big|_{r=1} = \sin\Big(\frac{\pi}{4} - \varphi\Big)\sin\theta = f(\varphi, \theta);$$

$$\text{(a)}\quad r > 1, \qquad \text{(b)}\quad r < 1.$$

Note that while the external Neumann problem does not require the fulfillment of the solvability condition (24.26), for the internal problem the following condition should be checked:

$$\int\limits_{r=1} u_n(r, \theta, \varphi)dS = \int\limits_{r=1} u_r(r, \theta, \varphi)dS = 0.$$

Indeed,

$$\int\limits_0^{2\pi}\int\limits_0^{\pi} \sin\Big(\frac{\pi}{4} - \varphi\Big)\sin^2\theta\, d\theta d\varphi = 0.$$

Thus, the internal Neumann problem is solvable.

Seek a solution of this problem by separation of variables in the form

$$u(r, \theta, \varphi) = R(r)Y(\theta, \varphi). \tag{24.62}$$

Since the function $u(r, \theta, \varphi)$ is harmonic outside the sphere of unit radius, then

$$|R(r)| < \infty, \quad |Y(\theta, \varphi)| < \infty, \qquad r > 1.$$

Upon separation of variables, we obtain for $R(r)$ the equation

$$r^2R'' + 2rR' - \lambda R = 0, \tag{24.63}$$

and for $Y(\varphi, \theta)$ the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial Y}{\partial\theta}\Big) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2} + \lambda Y = 0 \tag{24.64}$$

with the periodic boundary condition

$$Y(\theta, \varphi) = Y(\theta, \varphi + 2\pi).$$

We seek a solution of Eq. (24.64) in the form

$$Y(\theta, \varphi) = \Theta(\theta)\Phi(\varphi). \tag{24.65}$$

The function $\Phi(\varphi)$ is a solution of the Sturm–Liouville problem with a periodic boundary condition

$$\Phi'' + \mu\Phi = 0, \quad \Phi(\varphi + 2\pi) = \Phi(\varphi). \tag{24.66}$$

The solution of the problem (24.66) has the form

$$\Phi_m(\varphi) = C_m\cos m\varphi + D_m\sin m\varphi, \quad \mu = m^2, \quad m = \overline{0, \infty}. \tag{24.67}$$

In view of (24.67), for the function $\Theta(\theta)$ we have

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\Big(\sin\theta\frac{d\Theta}{d\theta}\Big)+\Big(\lambda-\frac{m^2}{\sin^2\theta}\Big)\Theta=0, \quad (24.68)$$
$$|\Theta(\theta)|<\infty, \qquad 0\leqslant\theta<\pi.$$

Perform in Eq. (24.67) the change of variables $z=\cos\theta$. Then for the function $\Theta(z)$ we get

$$\frac{d}{dz}\Big[(1-z^2)\frac{d\Theta}{dz}\Big]+\Big(\lambda-\frac{m^2}{1-z^2}\Big)\Theta=0, \quad (24.69)$$
$$|\Theta(z)|<\infty, \quad -1\leqslant z\leqslant 1.$$

The solution of the problem (24.69) is given by the associated Legendre functions (see Sec. "Associated Legendre functions" of Part III) with eigenvalues $\lambda=n(n+1)$:

$$\Theta_{nm}(z)=A_{nm}P_n^m(z)=A_{nm}(1-z^2)^{m/2}P_n^{(m)}(z), \quad n=\overline{0,\infty}.$$

Returning to the variable θ, we write

$$\Theta_{nm}(\theta)=A_{nm}\sin^m\theta\frac{d^m}{d(\cos\theta)^m}[P_n(\cos\theta)]=A_{nm}P_n^m(\cos\theta).$$

Since $P_n^m(\cos\theta)=0$ for $m>n$, then for the function (24.65) we get

$$Y_n^m(\theta,\varphi)=\bar C_{n0}P_n(\cos\theta)+ \\ +\sum_{m=1}^{n}(\bar C_{nm}\cos m\varphi+\bar D_{nm}\sin m\varphi)P_n^m(\cos\theta), \quad (24.70)$$

where

$$\bar C_{nm}=A_{nm}C_n, \qquad \bar D_{nm}A_{nm}D_n.$$

We seek a solution of Eq. (24.62) for $\lambda=n(n+1)$ in the form

$$R(r)=r^\alpha.$$

Then for α we obtain

$$\alpha(\alpha-1)+2\alpha-n(n+1)=0$$

or

$$\alpha^2+\alpha-n(n+1)=0.$$

Hence,

$$\alpha_1=n, \qquad \alpha_2=-(n+1)$$

and

$$R_n(r)=E_nr^n+F_nr^{-(n+1)}.$$

From the condition $|R(r)|<\infty$, $r>1$, we find $E_n=0$ for $n>1$, that is,

$$R_n(r)=\frac{F_n}{r^{n+1}}+E_0r^n\delta_{n0}, \quad (24.71)$$

where δ_{n0} is Kronecker's symbol. For the internal problem, we write analogously

$$R_n(r)=E_nr^n. \quad (24.72)$$

Substituting (24.70) and (24.71) into (24.62) and summing up over n, we get

$$u(r,\varphi,\theta)=\widetilde{E}_0+\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\Big[\widetilde{C}_{n0}P_n(\cos\theta)+$$
$$+\sum_{m=1}^{n}(\widetilde{C}_{nm}\cos m\varphi+\widetilde{D}_{nm}\sin m\varphi)P_n^m(\cos\theta)\Big], \tag{24.73}$$

where

$$\widetilde{C}_{nm}=F_n\bar{C}_{nm},\quad \widetilde{D}_{nm}=F_n\bar{D}_{nm},\quad \widetilde{E}_0=E_0\bar{C}_{00}.$$

Substitution of (24.73) into the boundary condition (24.61) yields

$$\frac{\partial u}{\partial r}\Big|_{r=1}=-\sum_{n=0}^{\infty}\frac{n+1}{r^{n+2}}\Big[\widetilde{C}_{n0}P_n(\cos\theta)+$$
$$+\sum_{m=1}^{n}(\widetilde{C}_{nm}\cos m\varphi+\widetilde{D}_{nm}\sin m\varphi)P_n^m(\cos\theta)\Big]\Big|_{r=1}=\sin\Big(\frac{\pi}{4}-\varphi\Big)\sin\theta. \tag{24.74}$$

Since $P_0(\cos\theta)=1$, Eq. (24.74) can be written

$$-\widetilde{C}_{00}-\sum_{n=1}^{\infty}\sum_{m=1}^{n}(n+1)(\widetilde{C}_{nm}\cos m\varphi+\widetilde{D}_{nm}\sin m\varphi)P_n^m(\cos\theta)=\sin\Big(\frac{\pi}{4}-\varphi\Big)\sin\theta.$$

Expand the right side in a series in orthogonal functions $\cos m\varphi P_n^m(\cos\theta)$ and $\sin m\varphi P_n^m(\cos\theta)$. Then for the coefficients $\widetilde{C}_{nm}$ and $\widetilde{D}_{nm}$ we obtain

$$-\widetilde{C}_{00}=\frac{1}{4\pi}\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\Big(\frac{\pi}{4}-\varphi\Big)\sin^2\theta\,d\theta,$$

$$-(n+1)\widetilde{C}_{nm}=\frac{(2n+1)(n-m)!}{2\pi(n+m)!}\int_0^{2\pi}\int_0^{\pi}\sin\Big(\frac{\pi}{4}-\varphi\Big)\sin\theta P_n^m(\cos\theta)\cos m\varphi\sin\theta\,d\theta d\varphi,$$

$$-(n+1)\widetilde{D}_{nm}=\frac{(2n+1)(n-m)!}{2\pi(n+m)!}\int_0^{2\pi}\int_0^{\pi}\sin\Big(\frac{\pi}{4}-\varphi\Big)\sin\theta P_n^m(\cos\theta)\sin m\varphi\sin\theta\,d\theta d\varphi,$$

where $m=\overline{1,n}$, $n=\overline{1,\infty}$. Since

$$\sin\Big(\frac{\pi}{4}-\varphi\Big)=\sin\frac{\pi}{4}\cos\varphi-\cos\frac{\pi}{4}\sin\varphi,\quad \sin\theta=P_1^1(\cos\theta),$$

then

$$\widetilde{C}_{00}=0,\quad \widetilde{C}_{nm}=\begin{cases}-\dfrac{\sin\pi/4}{2}, & n=m=1,\\ 0, & n\neq 1,\ m\neq 1;\end{cases}$$

$$\widetilde{D}_{nm}=\begin{cases}\dfrac{\cos\pi/4}{2}, & n=m=1,\\ 0, & n\neq 1,\ m\neq 1.\end{cases}$$

Finally,

$$u(r,\varphi,\theta) = \widetilde{E}_0 - \frac{1}{2r^2}\Big[\sin\frac{\pi}{4}\cos\varphi P_1^1(\cos\theta) - \cos\frac{\pi}{4}\sin\varphi P_1^1(\cos\theta)\Big]$$

or

$$u(r,\varphi,\theta) = \widetilde{E}_0 - \frac{1}{2r^2}\sin\Big(\frac{\pi}{4} - \varphi\Big)\sin\theta.$$

The constant $\widetilde{E}_0$ is arbitrary. If the function $u(r,\varphi,\theta)$ is subject to the regularity condition at infinity ($u = 0$ for $r \to \infty$), then $\widetilde{E}_0 = 0$ and the solution of the problem becomes unique.

For the internal Neumann problem, after similar manipulations with (24.72), we obtain

$$u(r,\varphi,\theta) = \widetilde{E}_0 + r\sin\Big(\frac{\pi}{4} - \varphi\Big)\sin\theta.$$

25 Poisson and Dini integrals

Theorem 25.1. *The solution of the first boundary value problem* (24.32) *for a circle*

$$\Delta_2 u = 0, \qquad u(r,\varphi)|_{r=a} = f(\varphi), \qquad r \leqslant a, \tag{25.1}$$

can be represented in the form

$$u(r,\varphi) = \frac{1}{2\pi}\int\limits_0^{2\pi} d\psi \frac{f(\psi)(a^2 - r^2)}{r^2 + a^2 - 2ar\cos(\psi - \varphi)}. \tag{25.2}$$

♦ *Relation* (25.2) *is called the Poisson integral of the first boundary value problem and the function*

$$\mathcal{K}(r,\varphi,a,\psi) = \frac{a^2 - r^2}{r^2 + a^2 - 2ar\cos(\psi - \varphi)} \tag{25.3}$$

is called the kernel of the Poisson integral of the first boundary value problem.

Proof. Substitute α_n and β_n from (24.46) into (24.47) and interchange the integration and the summation to get

$$u(r,\varphi) = \frac{1}{\pi}\int\limits_0^{2\pi} d\psi\, f(\psi)\Big\{\frac{1}{2} + \sum_{n=1}^{\infty}\Big(\frac{r}{a}\Big)^n[\cos n\varphi\cos n\psi + \sin n\varphi\sin n\psi]\Big\} =$$

$$= \frac{1}{\pi}\int\limits_0^{2\pi} d\psi\, f(\psi)\Big\{\frac{1}{2} + \sum_{n=1}^{\infty}\Big(\frac{r}{a}\Big)^n\cos n(\psi - \varphi)\Big\}.$$

However, for $|t| < 1$ the relation

$$\frac{1}{2} + \sum_{n=1}^{\infty} t^n\cos n\alpha = \frac{1}{2}\Big[1 + \sum_{n=1}^{\infty} t^n(e^{i\alpha n} + e^{-i\alpha n})\Big] =$$

$$= \frac{1}{2}\Big(1 + \frac{te^{i\alpha}}{1 - te^{i\alpha}} + \frac{te^{-i\alpha}}{1 - te^{-i\alpha}}\Big] = \frac{1 - t^2}{2(1 - 2t\cos\alpha + t^2)},$$

is valid, that is,

$$\frac{1}{2} + \sum_{n=1}^{\infty} t^n \cos n\alpha = \frac{1-t^2}{2(1-2t\cos\alpha + t^2)}, \qquad |t| < 1. \tag{25.4}$$

Now substitute $t = r/a$, $\alpha = \psi - \varphi$ into (25.4) to get (25.2). This proves the theorem.

◊ The solution of the external boundary value problem can be derived from (25.2) by using the change $a \to r$ and $r \to a$ (i.e., simply by changing the sign).

Theorem 25.2. *The solution of the second boundary value problem* (24.55) *for a circle* $r < a$ *can be represented by the formula*

$$u(r,\varphi) = C - \frac{a}{2\pi} \int\limits_{-\pi}^{\pi} f(\psi) \ln[a^2 + r^2 - 2ar\cos(\psi - \varphi)] d\psi, \tag{25.5}$$

called by the Dini formula or integral.

Proof. Substitute the values of the coefficients α_n and β_n into expression (24.55). Interchanging the summation and the integration yields

$$u(r,\varphi) = C + \frac{a}{\pi} \int\limits_{-\pi}^{\pi} f(\psi) \sum_{n=1}^{\infty} \frac{1}{n} \Big(\frac{r}{a}\Big)^n \cos n(\psi - \varphi)\, d\psi =$$

$$= C + \frac{a}{\pi} \int\limits_{-\pi}^{\pi} f(\psi) \operatorname{Re} \sum_{n=1}^{\infty} \frac{1}{n} \Big(\frac{r}{a}\Big)^n e^{in(\psi-\varphi)} d\psi = C + \frac{a}{\pi} \int\limits_{-\pi}^{\pi} f(\psi) \operatorname{Re} \sum_{n=1}^{\infty} \frac{t^n}{n} d\psi, \tag{25.6}$$

where $t = (r/a)^n e^{in\alpha}$, $\alpha = \psi - \varphi$, $|t| < 1$.

Using the known expansion

$$\ln \frac{1}{1-t} = \sum_{n=1}^{\infty} \frac{t^n}{n}, \qquad |t| < 1$$

and taking into account that

$$\operatorname{Re} \ln(1-t) = \ln|1-t| = \ln\Big|1 - \Big(\frac{r}{a}\Big)^n e^{in\alpha}\Big| = \frac{1}{2} \ln \frac{a^2 + r^2 - 2ar\cos\alpha}{a},$$

we find

$$u(r,\varphi) = C - \frac{a}{2\pi} \int\limits_{-\pi}^{\pi} f(\psi) \ln \frac{a^2 + r^2 - 2ar\cos\alpha}{a} d\psi,$$

and, taking into account the solvability condition for $f(\psi)$

$$u(r,\varphi) = C - \frac{a}{2\pi} \int\limits_{-\pi}^{\pi} f(\psi) \ln[a^2 + r^2 - 2ar\cos(\psi - \varphi)]\, d\psi,$$

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The summation in the series (24.57) is performed in a similar way giving the solution of the external boundary value problem for a circle.

26 Separation of variables in Helmholtz's equation

26.1 Separation of variables in Helmholtz's equation in polar coordinates

Let us consider Helmholtz's equation for a circle

$$\Delta u \pm k^2 u = 0 \tag{26.1}$$

with boundary conditions of the form

$$\text{(a) } u|_{r=a} = f(\varphi) \quad \text{or} \quad \text{(b) } \frac{\partial u}{\partial r}\Big|_{r=a} = g(\varphi). \tag{26.2}$$

Let $\Delta u + k^2 u = 0$. In polar coordinates, this equation takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + k^2 u = 0. \tag{26.3}$$

We seek a solution of Eq. (26.3) by separation of variables in the form:

$$u(r, \varphi) = R(r)\Phi(\varphi).$$

We obtain

$$\frac{\Phi}{r}\frac{d}{dr}(rR') + \frac{R}{r^2}\Phi'' + k^2\Phi R = 0.$$

By multiplying both sides by $r^2/(R\Phi)$ and separating the variables, we find

$$\frac{r}{R}\frac{d}{dr}(rR') + k^2 r^2 = -\frac{\Phi''}{\Phi} = \lambda.$$

Hence,

$$r^2 R'' + rR' + (k^2 r^2 - \lambda)R = 0 \tag{26.4}$$

and

$$\Phi'' + \lambda\Phi = 0 \tag{26.5}$$

with the conditions $\Phi(\varphi + 2\pi) = \Phi(\varphi)$ and $|R(r)| < \infty$ that follow from the problem statement. The Sturm–Liouville problem (26.5) was already considered (see example III.2.5), and its solution is

$$\lambda_n = n^2, \qquad \Phi_n(\varphi) = A_n \cos n\varphi + B_n \sin n\varphi. \tag{26.6}$$

Equation (26.4) with $\lambda = n^2$ is Bessel's equation whose bounded solution for $r < a$ has the form

$$R_n(r) = C_n J_n(kr). \tag{26.7}$$

As a result, the solutions of the original equation are given by

$$u(r, \varphi) = \sum_{n=0}^{\infty} J_n(kr)[A_n \cos n\varphi + B_n \sin n\varphi]. \tag{26.8}$$

Determine the coefficients A_n and B_n from boundary conditions:

(a) For boundary conditions of the first kind, expanding $f(\varphi)$ in a Fourier series, we find

$$u|_{r=a} = f(\varphi) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}(\alpha_n \cos n\varphi + \beta_n \sin n\varphi).$$

Hence, assuming that ka is not a root of a Bessel function [i.e., $J_m(ka) \neq 0$, $m = \overline{0, \infty}$], we obtain

$$A_0 2J_0(ka) = \alpha_0, \quad A_n J_n(ka) = \alpha_n, \quad B_n J_n(ka) = \beta_n, \tag{26.9}$$

where α_n and β_n are determined by formula (24.46) and the solution of the problem is unique and has the form

$$u(r, \varphi) = \frac{\alpha_0}{2}\frac{J_0(kr)}{J_0(ka)} + \sum_{n=1}^{\infty}\frac{J_n(kr)}{J_n(ka)}[\alpha_n \cos n\varphi + \beta_n \sin n\varphi]. \tag{26.10}$$

If ka is a root of a Bessel function [$J_m(ka) = 0$, $m = \overline{0, \infty}$], then the solution exists only if the coefficients α_n and β_n are equal to zero for $n = m$. In this case, the solution is not unique since the coefficients A_m and B_m cannot be determined from relation (26.9). If only one of the coefficients, α_n or β_n, is nonzero for $n = m$, the problem has no solution.

(b) Similarly, for boundary conditions of the second kind, we find

$$\frac{\partial u}{\partial r}\Big|_{r=a} = g(\varphi) = \frac{\bar{\alpha}_0}{2} + \sum_{n=1}^{\infty}(\bar{\alpha}_n \cos n\varphi + \bar{\beta}_n \sin n\varphi),$$

where $\bar{\alpha}_n$ and $\bar{\beta}_n$ are determined by relation (24.46) in which the functions $\varphi(x)$ and $\psi(x)$ should be replaced by the functions $\bar{\varphi}(x)$ and $\bar{\psi}(x)$. Thus, on the assumption that ka is not a root of the derivative of a Bessel function $J_n'(x)$, $n = \overline{0, \infty}$, we obtain

$$A_0 = \frac{\bar{\alpha}_0}{2kJ_0'(ka)}, \quad A_n = \frac{\bar{\alpha}_n}{kJ_n'(ka)}, \quad B_n = \frac{\bar{\beta}_n}{kJ_n'(ka)},$$

and the solution of the problem is unique and has the form

$$u(r, \varphi) = \frac{\bar{\alpha}_0}{2}\frac{J_0(kr)}{kJ_0'(ka)} \sum_{n=1}^{\infty}\frac{J_n(kr)}{kJ_n'(ka)}[\bar{\alpha}_n \cos n\varphi + \bar{\beta}_n \sin n\varphi], \tag{26.11}$$

otherwise, as for boundary conditions of the first kind, the problem has no solution or its solution is not unique.

If $\Delta u - k^2 u = 0$, then, instead of formulas (26.10) and (26.11), we obtain, respectively,

$$u(r, \varphi) = \frac{\alpha_0}{2}\frac{I_0(kr)}{I_0(ka)} + \sum_{n=1}^{\infty}\frac{I_n(kr)}{I_n(ka)}[\alpha_n \cos n\varphi + \beta_n \sin n\varphi], \tag{26.12}$$

$$u(r, \varphi) = \frac{\bar{\alpha}_0}{2}\frac{I_0(kr)}{kI_0'(ka)} + \sum_{n=1}^{\infty}\frac{I_n(kr)}{kI_n'(ka)}[\bar{\alpha}_n \cos n\varphi + \bar{\beta}_n \sin n\varphi], \tag{26.13}$$

where $I_n(x)$ is a modified Bessel function of the first kind. In this case, the solution exists for any value of ka and is unique since the Sturm–Liouville problem $\Delta u - \lambda u = 0$ has no positive eigenvalue and, hence, the value of k^2 cannot coincide with some eigenvalue. Note that this conclusion also follows immediately from formulas (26.12) and (26.13) since the modified Bessel function of the first kind $I_n(x)$ and its derivative have no zeros (see Sec. "The Sturm–Liouville problem for the Bessel equation" of Part III).

26.2 Separation of variables in the Helmholtz's equation in Cartesian coordinates

Let us consider Helmholtz's equation

$$\Delta u - k^2 u = 0, \qquad 0 \leqslant x \leqslant p, \quad 0 \leqslant y \leqslant q, \tag{26.14}$$

with boundary conditions

$$\text{(a)} \quad u|_{y=0} = \varphi(x), \quad u|_{y=q} = \psi(x), \quad u|_{x=0} = u|_{x=p} = 0 \tag{26.15}$$

or

$$\text{(b)} \quad \frac{\partial u}{\partial y}\Big|_{y=0} = \bar\varphi(x), \qquad \frac{\partial u}{\partial y}\Big|_{y=q} = \bar\psi(x), \qquad \frac{\partial u}{\partial x}\Big|_{x=0} = \frac{\partial u}{\partial x}\Big|_{x=p} = 0. \tag{26.16}$$

Problems with such boundary conditions arise upon reduction (see Sec. "Separation of variables in Laplace's equation") of a Neumann problem (provided that it is solvable) and a Dirichlet problem for Helmholtz's equation with arbitrary boundary conditions.

Seek a solution of Eq. (26.14) by the Fourier method in the form

$$u(x,y) = X(x)Y(y). \tag{26.17}$$

Substitute (26.17) into (26.14) to get

$$YX'' + XY'' - k^2 XY = 0.$$

Multiplication of both sides by $X(x)Y(y)$ and separation of variables yields

$$\frac{X''}{X} = -\frac{Y''}{Y} + k^2 = \lambda$$

or

$$X'' - \lambda X = 0, \qquad Y'' + (-k^2 + \lambda)Y = 0. \tag{26.18}$$

From the original problem for the first equation of (26.18) we obtain the Sturm–Liouville problem

$$X(0) = X(p) = 0, \qquad |X(x)| < \infty$$

whose solution has the form (see example III.2.2)

$$\lambda = -\Big(\frac{\pi n}{p}\Big)^2 = -\omega_n^2, \qquad X_n(x) = C_n \sin\frac{\pi n x}{p}. \tag{26.19}$$

Substitute λ_n into (26.18) to get

$$Y_n(y) = A_n e^{\sqrt{\omega_n^2+k^2}y} + B_n e^{-\sqrt{\omega_n^2+k^2}y} = \widetilde{A}_n \coth(\sqrt{\omega_n^2+k^2}y) + \widetilde{B}_n \operatorname{sh}(\sqrt{\omega_n^2+k^2}y).$$

Thus,

$$u(x,y) = \sum_{n=1}^{\infty} [\bar A_n \operatorname{ch}(\sqrt{\omega_n^2+k^2}y) + \bar B_n \operatorname{sh}(\sqrt{\omega_n^2+k^2}y)] \sin\omega_n x,$$

where $\bar A_n = \widetilde{A}_n C_n$, $\bar B_n = \widetilde{B}_n C_n$.

(a) From the boundary conditions of the first kind we find

$$u|_{y=0} = \sum_{n=1}^{\infty} \alpha_n \sin\frac{\pi n x}{p}, \qquad u|_{y=q} = \sum_{n=1}^{\infty} \beta_n \sin\frac{\pi n x}{p},$$

where

$$\alpha_n = \frac{2}{p}\int\limits_0^p \varphi(x)\sin\omega_n x\,dx, \qquad \beta_n = \frac{2}{p}\int\limits_0^p \psi(x)\sin\omega_n x\,dx.$$

Hence, for the coefficients $\bar{A}_n$ and $\bar{B}_n$, we have the system of algebraic equations

$$\begin{aligned}&\bar{A}_n = \alpha_n,\\ &\bar{A}_n \coth(\sqrt{\omega_n^2+k^2}q) + \bar{B}_n \sinh(\sqrt{\omega_n^2+k^2}q) = \beta_n.\end{aligned}$$

Then

$$\begin{aligned}\bar{A}_n &= \alpha_n,\\ \bar{B}_n &= \frac{1}{\sinh(\sqrt{\omega_n^2+k^2}q)}\big\{\beta_n - \alpha_n\coth(\sqrt{\omega_n^2+k^2}q)\big\}.\end{aligned}$$

Finally,

$$u(x,y) = \sum_{n=1}^{\infty}\Big\{\alpha_n\coth(\sqrt{\omega_n^2+k^2}y) - \\ -\frac{\sinh(\sqrt{\omega_n^2+k^2}y)}{\sinh(\sqrt{\omega_n^2+k^2}q)}\big[\beta_n - \alpha_n\coth(\sqrt{\omega_n^2+k^2}q)\big]\Big\}\sin\omega_n x.$$

The solution of the problem (26.16) can be found analogously.

◊ Equation $\Delta u + k^2 u = 0$ is worked in a similar way. It should only be checked whether the value of k^2 coincides with some of the eigenvalues of the Sturm–Liouville problem (see the preceding section).

26.3 Separation of variables in Helmholtz's equation in spherical coordinates

Let us turn to an example where the solution of Helmholtz's equation in spherical coordinates can be found by separation of variables. Let us restrict ourselves to the case where the boundary conditions are independent of the variable φ. In the general case, the solution is similar to that obtained in example 24.7.

Example 26.1. Find a solution of Helmholtz's equation for a sphere:

$$\Delta u + 25u = 0, \qquad \frac{\partial u}{\partial r}\Big|_{r=\pi} = \cos\theta, \quad 0 \leqslant r < \pi.$$

Solution. In spherical coordinates, Helmholtz's equation takes the form

$$\Delta_3 u + 25u = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} + 25u = 0. \quad (26.20)$$

Neither the equation nor the boundary condition involves the variable φ; therefore, we assume that $\partial u/\partial\varphi = 0$. Seek a solution by the Fourier method in the form

$$u(r,\theta) = R(r)\Theta(\theta). \qquad (26.21)$$

Denote

$$R' = \frac{dR}{dr} \quad\text{and}\quad \dot{\Theta} = \frac{d\Theta}{d\theta}.$$

Substitute (26.21) into (26.20) to get

$$\frac{\Theta}{r^2}\frac{d}{dr}\Big(r^2\frac{dR}{dr}\Big) + \frac{R}{r^2\sin\theta}\frac{d}{d\theta}\Big(\sin\theta\frac{d\Theta}{\partial\theta}\Big) + 25R\Theta = 0$$

or

$$\frac{\Theta}{r^2}(2rR' + r^2R'') + \frac{R}{r^2\sin\theta}(\cos\theta\dot{\Theta} + \sin\theta\ddot{\Theta}) + 25R\Theta = 0.$$

Multiply both sides by $r^2/\Theta R$ and separate the variables to get

$$\frac{2rR' + r^2R''}{R} + 25r^2 = -\frac{1}{\Theta\sin\theta}(\cos\theta\dot{\Theta} + \sin\theta\ddot{\Theta}) = \lambda.$$

As a result, the function $R(r)$ is determined from the equation

$$r^2R'' + 2rR' + (25r^2 - \lambda)R = 0, \qquad 0 \leqslant r \leqslant \pi, \quad |r| < \infty,$$

and the function $\Theta(\theta)$ is a solution of the Sturm–Liouville problem

$$\ddot{\Theta} + \dot{\Theta}\operatorname{ctg}\theta + \lambda\Theta = 0, \qquad 0 \leqslant \theta \leqslant \pi, \quad |\Theta(\theta)| < \infty. \tag{26.22}$$

The solution of the Sturm–Liouville problem (26.22) is

$$\lambda_n = n(n+1), \qquad \Theta_n(\theta) = A_nP_n(\cos\theta), \quad n = \overline{0,\infty}$$

(see example III.15.1). Substitute λ_n into the equation for the function $R(r)$ to get

$$R'' + \frac{2}{r}R' + \Big[25 - \frac{n(n+1)}{r^2}\Big]R = 0.$$

We seek a solution of this equation in the form

$$R(r) = z(r)r^{-1/2}.$$

Then

$$R'(r) = r^{-1/2}z'(r) - \frac{1}{2}r^{-3/2}z(r);$$

$$R''(r) = r^{-1/2}z''(r) - r^{-3/2}z'(r) + \frac{3}{4}r^{-5/2}z(r).$$

For the function $z(r)$, we obtain the equation

$$r^2z'' + rz' + \Big[(5r)^2 - \Big(n + \frac{1}{2}\Big)^2\Big]z = 0. \tag{26.23}$$

The general solution of Eq. (26.23) has the form

$$z_n(r) = B_nJ_{n+1/2}(5r) + C_nJ_{-n-1/2}(5r),$$

where B_n and C_n are arbitrary constants.

For the function $R_n(r)$ we obtain

$$R_n(r) = r^{-1/2}[B_nJ_{n+1/2}(5r) + C_nN_{n+1/2}(5r)]. \tag{26.24}$$

In view of the well-known property of the Neumann functions that

$$\lim_{x\to+0} N_{n+1/2}(x) = -\infty,$$

we have $C_n = 0$ and

$$R_n(r) = B_nr^{-1/2}J_{n+1/2}(5r).$$

Then the solution is written as

$$u_n(r,\theta) = R_n(r)\Theta_n(\theta) = A_n P_n(\cos\theta)\frac{B_n}{\sqrt{r}} J_{n+1/2}(5r).$$

Summing up this expression over n, we find

$$u(r,\theta) = \sum_{n=0}^{\infty} \bar{A}_n P_n(\cos\theta) r^{-1/2} J_{n+1/2}(5r), \tag{26.25}$$

where $\bar{A}_n = A_n B_n$. From the boundary condition

$$\frac{\partial u}{\partial r}\Big|_{r=\pi} = \cos\theta, \tag{26.26}$$

we calculate the constants $\bar{A}_n$. To do this, differentiate $u(r,\theta)$ (26.25) with respect to r:

$$\frac{\partial u}{\partial r}\Big|_{r=\pi} = \sum_{n=0}^{\infty} \bar{A}_n P_n(\cos\theta)\Big[5J'_{n+1/2}(5r)r^{-1/2} - \frac{1}{2}r^{-3/2}J_{n+1/2}(5r)\Big]\Big|_{r=\pi} =$$

$$= \sum_{n=0}^{\infty} \bar{A}_n P_n(\cos\theta)\Big[5J'_{n+1/2}(5\pi)\frac{1}{\sqrt{\pi}} - \frac{1}{2}\frac{1}{\sqrt{\pi^3}}J_{n+1/2}(5\pi)\Big]. \tag{26.27}$$

Expand $\cos\theta$ in a Fourier series of Legendre polynomials. The function $\cos\theta$ is a first degree polynomial in $\cos\theta$. Hence, the Fourier series of this function will not contain polynomials of higher than the first degree. Since

$$P_0(x) = 1, \qquad P_1(x) = x,$$

then

$$\cos\theta = 0\cdot P_0(\cos\theta) + P_1(\cos\theta) + \sum_{n=2}^{\infty} 0\cdot P_n(\cos\theta). \tag{26.28}$$

Hence,

$$\alpha_n = \delta_{n1}$$

Substituting (26.28) and (26.27) into (26.26) and equating the coefficients of identical polynomials $P_n(\cos\theta)$, we obtain

$$\bar{A}_n = \alpha_n \Big/ \Big[5J'_{n+1/2}(5r)\frac{1}{\sqrt{\pi}} - \frac{1}{2}\frac{1}{\sqrt{\pi^3}}J_{n+1/2}(5r)\Big], \quad n = \overline{0,\infty},$$

or

$$\bar{A}_n = \delta_{n1} \Big/ \Big[5J'_{n+1/2}(5\pi)\frac{1}{\sqrt{\pi}} - \frac{1}{2}\frac{1}{\sqrt{\pi^3}}J_{n+1/2}(5\pi)\Big].$$

Then

$$u(r,\theta) = \frac{2(\sqrt{\pi})^3}{\Big[10\pi J'_{3/2}(5\pi) - J_{3/2}(5\pi)\Big]} P_1(\cos\theta)\frac{J_{3/2}(5r)}{\sqrt{r}}.$$

In view of relations (III.5.13), we get

$$u(r,\theta) = -\frac{\pi^2}{2r}\Big(\frac{\sin 5r}{5r} - \cos 5r\Big)\cos\theta.$$

27 The Green's function method

In preceding sections we have considered the Fourier method that allows us to represent a solution of a boundary value problem as a Fourier series of the eigenvalues of the corresponding Sturm–Liouville problem. A solution of a boundary value problem can also be found in integral form using the Green's function method. Before proceeding to the consideration of this method, we should note that in Sec. "Poisson and Dini integrals", the summation of the Fourier series has resulted in Poisson's formula (25.2) and Dini's formula (25.5). In integral form, these formulas represent the solution of the Dirichlet and Neumann problems for a circle. In this section, we shall obtain the same formulas by the Green's function method.

27.1 The Green's function of the Dirichlet problem

Let us turn to the Dirichlet problem for Poisson's equation for a region E confined by a surface S_E

$$\Delta u(\vec{x}) = -F(\vec{x}), \qquad \vec{x} \in E \subset \mathbb{R}^n, \quad n = 2, 3; \tag{27.1}$$

$$u(\vec{x}) = f(\vec{x}), \qquad \vec{x} \in S_E. \tag{27.2}$$

This problem can be considered, for instance, as the problem of finding the electrostatic (gravitational, thermal, etc.) potential created by charges distributed in a region E with density $\rho(x) = F(\vec{x})/(4\pi)$ (see the notation in Sec. "Stationary physical properties and elliptic equations"). The potential value on the boundary S_E is given. Generalizing the well-known approach (see Sec. "A boundary value problem with a parameter" of Part II) to the solution of boundary value problems for ordinary differential equations by the Green's function method, let us formulate the following auxiliary problem: Find the potential created by a point charge of magnitude $1/(4\pi)$ within a region confined by a grounded conducting sphere of surface S_E.

Denote the sought-for potential by $G(\vec{x}, \vec{y})$. In contrast to $u(\vec{x})$, the function $G(\vec{x}, \vec{y})$ depends not only on the variable $\vec{x}$, but also on the parameter $\vec{y}$ that specifies the position of the point charge. Since the point charge density can be written, using the delta function, as $-\delta(\vec{x} - \vec{y})/(4\pi)$ and the conducting sphere is grounded, we arrive at the following boundary value problem:

$$\Delta_x G(\vec{x}, \vec{y}) = -\delta(\vec{x} - \vec{y}), \qquad \vec{x}, \vec{y} \in E,$$

$$G(\vec{x}, \vec{y})\big|_{\vec{x} \in S_E} = 0.$$

Here and below, the subscript of the Laplace operator indicates the variable upon which it acts.

The essence of the Green's function method is that the solution of the general problem (27.1), (27.2) is represented in terms of the solution of a special kind boundary value problem, that is, in terms of the function $G(\vec{x}, \vec{y})$, called Green's function or a source function.

♦ The *Green's function* (*source function*) of the Dirichlet problem (27.1), (27.2) is the function $G(\vec{x}, \vec{y})$ that is the solution of the boundary value problem

$$\Delta_x G(\vec{x}, \vec{y}) = -\delta(\vec{x} - \vec{y}), \quad \vec{x}, \vec{y} \in E \subset \mathbb{R}^n, \ n = 2, 3; \tag{27.3}$$

$$G(\vec{x}, \vec{y})\big|_{\vec{x} \in S_E} = 0. \tag{27.4}$$

From this definition it follows, first, that the Green's function $G(\vec{x},\vec{y})$, being the solution of a nonhomogeneous equation, Eq. (27.3), admits the representation

$$G(\vec{x},\vec{y}) = g_0(\vec{x},\vec{y}) - \mathcal{E}_n(\vec{x},\vec{y}), \tag{27.5}$$

where $g_0(\vec{x},\vec{y})$ is a regular harmonic function and $\mathcal{E}_n(\vec{x},\vec{y})$ is the fundamental solution of the Laplace equation, that is,

$$\Delta_x g_0(\vec{x},\vec{y}) = 0, \tag{27.6}$$

$$\Delta_x \mathcal{E}_n(\vec{x},\vec{y}) = \delta(\vec{x}-\vec{y}). \tag{27.7}$$

Second, the Green's function $G(\vec{x},\vec{y})$ possesses the property of symmetry

$$G(\vec{x},\vec{y}) = G(\vec{y},\vec{x}). \tag{27.8}$$

Actually, put in the second Green's function (21.2)

$$u(\vec{x}) = G(\vec{x},\vec{y}), \qquad v(\vec{x}) = G(\vec{x},\vec{z}).$$

Then

$$\int\limits_E [G(\vec{x},\vec{y})\Delta_x G(\vec{x},\vec{z}) - G(\vec{x},\vec{z})\Delta_x G(\vec{x},\vec{y})]\,dx =$$

$$= \int\limits_{S_E} \big([G(\vec{x},\vec{y})\nabla_x G(\vec{x},\vec{z}) - G(\vec{x},\vec{z})\nabla_x G(\vec{x},\vec{y})], d\vec{S}\big).$$

From here, in view of (27.3), (27.4), we obtain

$$\int\limits_E [G(\vec{x},\vec{y})\delta(\vec{x}-\vec{z}) + G(\vec{x},\vec{z})\delta(\vec{x}-\vec{y})]\,dx = -G(\vec{z},\vec{y}) + G(\vec{y},\vec{z}) = 0$$

and, respectively, (27.8). In view of the equality $\mathcal{E}_n(\vec{x},\vec{y}) = \mathcal{E}_n(|\vec{x}-\vec{y}|)$ that signifies the symmetry of the fundamental solution with respect to the variables $\vec{x}$ and $\vec{y}$ [i.e., $\mathcal{E}_n(\vec{x},\vec{y}) = \mathcal{E}_n(\vec{y},\vec{x})$], from (27.8) it follows that the harmonic function

$$g_0(\vec{x},\vec{y}) = g_0(\vec{y},\vec{x}) \tag{27.9}$$

is symmetric as well. Therefore, not only (27.6) is valid, but

$$\Delta_y g_0(\vec{x},\vec{y}) = 0 \tag{27.10}$$

as well.

♢ Returning to the physical sense of the function $G(\vec{x},\vec{y})$, we can state that the principle of symmetry, formulated above for $G(\vec{x},\vec{y})$, is nothing but a mathematical expression of the principle of reversibility in physics, namely: a source placed at a point $\vec{y}$ produces at a point $\vec{x}$ the same action as a source placed at the point $\vec{x}$ does at the point $\vec{y}$.

♢ The Green's function of the external Dirichlet problem (for infinite regions) is determined analogously [Eqs. (27.3), (27.4)] with the additional condition of regularity on the function $G(\vec{x},\vec{y}) \to 0$ as $|\vec{x}| \to \infty$ and $\vec{y}$ is fixed. This condition is also necessary for internal problems with infinite surfaces, such as the Dirichlet problem for a semispace.

♢ If for Eq. (27.3) the boundary condition of the first kind (27.4) is replaced by a boundary condition of the third kind, we come to the definition of the Green's function of the third boundary value problem.

Now we turn to finding a solution of the problem (27.1), (27.2).

Theorem 27.1. *If the solution of the Dirichlet problem* (27.1), (27.2) *exists, it can be represented in the form*

$$u(\vec{x}) = \int\limits_E G(\vec{x},\vec{y})F(\vec{y})d\vec{y} - \int\limits_{S_E} f(\vec{y})\frac{\partial G(\vec{x},\vec{y})}{\partial n_y}dS_y. \tag{27.11}$$

Proof. If $u(\vec{x})$ and S_E satisfy the conditions of Lemma 23.1, then, in view of (23.2),

$$u(\vec{x}) = \int\limits_E \mathcal{E}_n(\vec{x},\vec{y})\Delta_y u(\vec{y})d\vec{y} + \oint\limits_{S_E}\Big[u(\vec{y})\frac{\partial \mathcal{E}_n(\vec{x},\vec{y})}{\partial n_y} - \mathcal{E}_n(\vec{x},\vec{y})\frac{\partial u(\vec{y})}{\partial n_y}\Big]dS_y. \tag{27.12}$$

For convenience, redenote $\vec{x}$ by $\vec{y}$ in the second Green's formula (21.2) and apply this formula to the functions $u(\vec{y})$ and $g_0(\vec{x},\vec{y})$, taking into account (27.2). Then

$$-\int\limits_E g_0(\vec{x},\vec{y})\Delta_y u(\vec{y})d\vec{y} - \oint\limits_{S_E}\Big[u(\vec{y})\frac{\partial g_0(\vec{x},\vec{y})}{\partial n_y} - g_0(\vec{x},\vec{y})\frac{\partial u(\vec{y})}{\partial n_y}\Big]dS_y = 0. \tag{27.13}$$

Combining (27.12) and (27.13), we have

$$u(\vec{x}) = \int\limits_E [\mathcal{E}_n(\vec{x},\vec{y}) - g_0(\vec{x},\vec{y})]\Delta_y u(\vec{y})d\vec{y} +$$

$$+\oint\limits_{S_E}\Big\{u(\vec{y})\frac{\partial}{\partial n_y}[\mathcal{E}_n(\vec{x},\vec{y}) - g_0(\vec{x},\vec{y})] - [\mathcal{E}_n(\vec{x},\vec{y}) - g_0(\vec{x},\vec{y})]\frac{\partial u(\vec{y})}{\partial n_y}\Big\}dS_y.$$

In view of (27.5), this can be written

$$u(\vec{x}) = -\int\limits_E G(\vec{x},\vec{y})\Delta_y u(\vec{y})d\vec{y} - \oint\limits_{S_E}\Big[u(\vec{y})\frac{\partial}{\partial n_y}G(\vec{x},\vec{y}) - G(\vec{x},\vec{y})\frac{\partial u(\vec{y})}{\partial n_y}\Big]dS_y \tag{27.14}$$

and simplified, in view of relations (27.1) and (27.2) for $u(\vec{x})$ and the boundary condition (27.4) for $G(\vec{x},\vec{y})$, to the relation

$$u(\vec{x}) = \int\limits_E G(\vec{x},\vec{y})F(\vec{y})d\vec{y} + \oint\limits_{S_E} f(\vec{y})\frac{\partial}{\partial n_y}G(\vec{x},\vec{y})\,dS_y$$

coinciding with (27.11).

As follows from the theorem proved, to solve the problem (27.1), (27.2), it suffices to find the function $G(\vec{x},\vec{y})$. Now we just proceed to investigating this issue.

27.2 Methods for construction of Green's function for a Dirichlet problem

There are several methods for the construction of the Green's function defined by (27.3), (27.4).

I. We begin with the generalization of the method of construction of the Green's function of a boundary value problem for an ordinary differential equation (see Sec. "The Green's function of a boundary value problem" of Part II), which is based on the use of the eigenfunctions of the corresponding Sturm–Liouville problem and of the expansion of the function $\delta(\vec{x}-\vec{y})$ over these functions (see Sec. "The Dirac delta function and systems of orthonormalized functions" of Part II).

Thus, we consider, along with (27.3), (27.4), the Sturm–Liouville problem

$$\Delta v(\vec{x}) + \lambda v(\vec{x}) = 0, \qquad \vec{x} \in E \subset \mathbb{R}^k, \quad k = 2, 3; \tag{27.15}$$
$$v(\vec{x})\Big|_{\vec{x}\in S_E} = 0.$$

Let $v_n(\vec{x})/\|v_n(x)\|$ be a complete orthonormalized system of the eigenfunctions of this problem with eigenvalues λ_n, $n = \overline{1, \infty}$. We seek the function $G(\vec{x}, \vec{y})$ in the form of a series

$$G(\vec{x}, \vec{y}) = \sum_{n=1}^{\infty} A_n \frac{v_n(\vec{x})}{\|v_n\|}. \tag{27.16}$$

In this form, the function $G(\vec{x}, \vec{y})$ already satisfies the boundary condition (27.4). Substitution of (27.16) into Eq. (27.3), in view of (27.15) and the expansion

$$\delta(\vec{x} - \vec{y}) = \sum_{n=1}^{\infty} \frac{v_n(\vec{x})v_n(\vec{y})}{\|v_n\|^2},$$

yields

$$\sum_{n=1}^{\infty} \frac{A_n}{\|v_n\|} \Delta_x v_n(\vec{x}) = -\sum_{n=1}^{\infty} \frac{A_n \lambda_n v_n(\vec{x})}{\|v_n\|} = -\sum_{n=1}^{\infty} \frac{v_n(\vec{x})v_n(\vec{y})}{\|v_n\|^2},$$

whence, on condition that $\lambda_n \neq 0$ for $n = \overline{1, \infty}$, we obtain

$$A_n = \frac{v_n(\vec{y})}{\lambda_n \|v_n\|}.$$

Substitution of this into (27.16) yields

$$G(\vec{x}, \vec{y}) = \sum_{n=1}^{\infty} \frac{v_n(\vec{x})v_n(\vec{y})}{\lambda_n \|v_n\|^2}. \tag{27.17}$$

II. Let us consider next the method of electrostatic images (method of reflections). The idea of this method, which comes from the electrostatic interpretation of Green's function, is that for a given point charge and a surface S_E, additional charges are imaged in such a way that the net potential on the surface be equal to zero, as this is prescribed by the boundary condition (27.4).

Let us consider several examples of the construction of the Green's function in given regions $E \in \mathbb{R}^n$, $n = 2, 3$.

Example 27.1. Construct Green's function for a semispace (semiplane) $E : x_n > 0$, $S_E : x_n = 0$.

Solution. Assume that a point $\vec{y}$ lies in a semispace $x_n > 0$. Then the point $\vec{y}'$ symmetric to the point $\vec{y}$ about the plane $x_n = 0$ lies in the semispace $x_n < 0$. It can readily be seen (see Fig. 34 where the point $\vec{x}_0$ lies in the plane $x_n = 0$) that, as the point $\vec{y}$ approaches the plane $x_n = 0$, the symmetric point $\vec{y}'$ also approaches this plane, and these points merge together on the plane. If a charge equal in magnitude to the charge located at the point $\vec{y}$, but opposite in sign, is placed at the point $\vec{y}'$, the sought-for Green's function can be written in the form

$$G(\vec{x}, \vec{y}) = -\mathcal{E}_n(\vec{x}, \vec{y}) + \mathcal{E}_n(\vec{x}, \vec{y}'). \tag{27.18}$$

Indeed, let us act on (27.18) by the operator Δ_x to get

$$\Delta_{\vec{x}} G(\vec{x}, \vec{y}) = -\delta(\vec{x} - \vec{y}) + \delta(\vec{x} - \vec{y}\,').$$

From this relation it can be seen that the condition (27.3) is fulfilled since, in the region under consideration $x_n > 0$, the function $\delta(\vec{x} - \vec{y}\,') = 0$. Along with this, in view of the fact that the equality $|\vec{x} - \vec{y}| = |\vec{x} - \vec{y}\,'|$ holds immediately on the plane $x_n = 0$, we make sure that the boundary condition (27.4) is valid:

$$G(\vec{x}, \vec{y}) = -\mathcal{E}_n(\vec{x}, \vec{y}) + \mathcal{E}_n(\vec{x}, \vec{y}) = -\mathcal{E}_n(|\vec{x} - \vec{y}|) + \mathcal{E}_n(|\vec{x} - \vec{y}|) = 0,$$

since $\mathcal{E}_n(\vec{x}, \vec{y}) = \mathcal{E}_n(|\vec{x} - \vec{y}|)$.

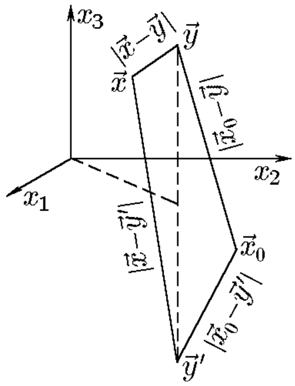

Fig. 34.

Finally, it can readily be verified that $G(\vec{x}, \vec{y}) \to 0$ as $|\vec{x}| \to \infty$. This becomes quite evident if we write the Green's function (27.18) using the explicit form of the fundamental solutions $\mathcal{E}_n(\vec{x}, \vec{y})$ of Laplace's equation

$$G(\vec{x}, \vec{y}) = \begin{cases} \dfrac{1}{4\pi}\Big(\dfrac{1}{|\vec{x} - \vec{y}|} - \dfrac{1}{|\vec{x} - \vec{y}\,'|}\Big), & n = 3; \\ \dfrac{1}{2\pi}\Big(\ln \dfrac{1}{|\vec{x} - \vec{y}|} - \ln \dfrac{1}{|\vec{x} - \vec{y}\,'|}\Big), & n = 2. \end{cases} \qquad (27.19)$$

Thus, formula (27.19) is indeed the solution of the problem (27.3), (27.4) for the region $x_n > 0$.

Example 27.2. Construct Green's function for the interior of a sphere (circle) E: $|\vec{x}| < R$, S_E: $|\vec{x}| = R$.

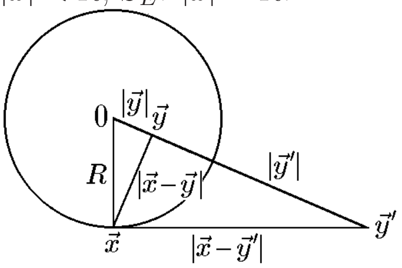

Fig. 35.

Solution. Let a point $\vec{y} \in E$, $\vec{y} \neq 0$, and $\vec{y}\,' = \vec{y}r^2/|\vec{y}|^2$ be the point symmetric to $\vec{y}$ about the sphere (Fig. 35), that is,

$$|\vec{y}|\,|\vec{y}\,'| = R^2. \qquad (27.20)$$

Assume that at the point $\vec{y}$ there is a charge equal to $1/(4\pi)$ and then place a charge of magnitude $-\alpha/(4\pi)$ to the point $\vec{y}\,'$. Since $\alpha\delta(\vec{x} - \vec{y}\,') = \delta([\vec{x} - \vec{y}\,']/\alpha)$, we seek Green's function in the form

$$G(\vec{x}, \vec{y}) = -\mathcal{E}_n(|\vec{x} - \vec{y}|) + \mathcal{E}_n(|\vec{x} - \vec{y}\,'|/\alpha). \qquad (27.21)$$

Let us choose a value of α such that the function $G(\vec{x}, \vec{y})$ vanish on the boundary S_E. Note that the triangles $O\vec{x}\vec{y}$ and $O\vec{x}\vec{y}\,'$ are similar: they have one angle in common and their adjacent sides, according to (27.20), are proportional. Therefore, on the surface S_E: $|\vec{x}| = R$ the relation

$$\frac{R}{|\vec{y}|} = \frac{|\vec{x} - \vec{y}\,'|}{|\vec{x} - \vec{y}|}$$

is valid, and, hence, in view of (27.21), we should put $\alpha = R/|\vec{y}|$. Then

$$G(\vec{x}, \vec{y}) = -\mathcal{E}_n(|\vec{x} - \vec{y}|) + \mathcal{E}_n(|\vec{y}|\,|\vec{x} - \vec{y}\,'|/R) = -\mathcal{E}_n(|\vec{x} - \vec{y}|) + \mathcal{E}_n\Big(\frac{|\vec{y}|}{R}\Big|\vec{x} - \vec{y}\frac{R^2}{|\vec{y}|^2}\Big|\Big). \qquad (27.22)$$

By virtue of (27.20), this formula is also valid for $\vec{y} = 0$:

$$G(\vec{x}, 0) = -\mathcal{E}_n(|\vec{x}|) + \mathcal{E}_n(R). \qquad (27.23)$$

As in the preceding example, one can readily verify that formulas (27.22) and (27.23) satisfy the problem (27.3), (27.4) and write out them in explicit form for $n = 2$ and $n = 3$.

Following the principle used in the above examples, Green's functions can be found for various regions: a dihedral angle, a sector, a semisphere, a semicircle, and the like.

III. Let us now consider a method which is a combination of those described above. We restrict ourselves to considering a three-dimensional spherical region E: $|\vec{x}| < R$, S_E: $|\vec{x}| = R$. The solution of the problem (27.3), (27.4) is sought in the form of (27.5). For $n = 3$ it becomes

$$G(\vec{x}, \vec{y}) = \frac{1}{4\pi|\vec{x} - \vec{y}|} + g_0(\vec{x}, \vec{y}). \tag{27.24}$$

Thus, the problem (27.3), (27.4) is reduced to the following

$$\Delta_x g_0(\vec{x}, \vec{y}) = 0, \qquad \vec{x} \in E; \tag{27.25}$$
$$g_0(\vec{x}, \vec{y}) = -\frac{1}{4\pi|\vec{x} - \vec{y}|}, \quad \vec{x} \in S_E.$$

We seek a solution of this problem by the method of separation of variables. Taking into account the geometry of the problem, we use a spherical coordinate system with the origin at the center of the sphere and the z-axis directed along the vector $\vec{y}$. With this choice of $\vec{x}(r, \varphi, \theta)$ and $\vec{y}(\rho, 0, 0)$, the boundary conditions are independent of φ. Following example 24.7, we find the function $g_0(\vec{x}, \vec{y})$ in the form

$$g_0(\vec{x}, \vec{y}) = \sum_{n=0}^{\infty} A_n \Big(\frac{r}{R}\Big)^n P_n(\cos\theta). \tag{27.26}$$

To determine the unknown coefficients A_n, substitute (27.26) into (27.25) to get

$$\sum_{n=0}^{\infty} A_n P_n(\cos\theta) = -\frac{1}{4\pi\sqrt{r^2 + \rho^2 - 2R\rho\cos\theta}} =$$
$$= -\frac{1}{4\pi R\sqrt{1 - 2(\rho/R)\cos\theta + (\rho/R)^2}}. \tag{27.27}$$

Since $\rho/R < 1$, the right side of this expression can be considered as a generating function of Legendre polynomials (see Sec. "Legendre polynomials" of Part III). As a result, formula (27.27) can be written

$$\sum_{n=0}^{\infty} A_n P_n(\cos\theta) = -\frac{1}{4\pi R}\sum_{n=0}^{\infty}\Big(\frac{\rho}{R}\Big)^n P_n(\cos\theta),$$

whence

$$A_n = -\frac{1}{4\pi R}\Big(\frac{\rho}{R}\Big)^n,$$

in view of which we find

$$g_0(\vec{x}, \vec{y}) = -\frac{1}{4\pi R}\sum_{n=0}^{\infty}\Big(\frac{\rho r}{R^2}\Big)^n P_n(\cos\theta).$$

The point $\vec{y}'$, symmetric to the point $\vec{y}$ about the sphere S_E, will have coordinates $(\rho', 0, 0)$, where $\rho' = R^2/\rho < R$. Then

$$g_0(\vec{x}, \vec{y}) = -\frac{1}{4\pi R}\sum_{n=0}^{\infty}\Big(\frac{r}{\rho'}\Big)^n P_n(\cos\theta).$$

Relation $r/\rho' < 1$ enables us to convolve this series to a generating function, that is, to get

$$g_0(\vec{x}, \vec{y}) = -\frac{1}{4\pi R}\frac{1}{\sqrt{1 - 2(r/\rho')\cos\theta + (r/\rho')^2}} =$$
$$= -\frac{\rho'}{4\pi R}\frac{1}{\sqrt{(\rho')^2 - 2\rho' r\cos\theta + r^2}} =$$
$$= -\frac{\rho'}{4\pi R|\vec{x} - \vec{y}'|} = -\frac{R}{4\pi\rho|\vec{x} - \vec{y}'|} = -\frac{R}{4\pi|\vec{y}|\,|\vec{x} - \vec{y}'|}.$$

Returning to (27.24), we find the explicit form of the sought-for Green's function

$$G(\vec{x}, \vec{y}) = \frac{1}{4\pi}\Big(\frac{1}{|\vec{x} - \vec{y}|} - \frac{R}{|\vec{y}|\,|\vec{x} - \vec{y}'|}\Big), \tag{27.28}$$

that coincides with (27.22) for $n = 3$.

CHAPTER 6

Hyperbolic equations

28 The Cauchy problem for the one-dimensional homogeneous wave equation. D'Alembert's formula

Let us consider a one-dimensional hyperbolic equation on an infinite interval, which describes free vibrations of an infinitely long string:

$$\begin{aligned} &u_{tt} - a^2 u_{xx} = 0, \quad -\infty < x < \infty, \\ &u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \end{aligned} \tag{28.1}$$

Theorem 28.1. *The solution of the Cauchy problem* (28.1) *has the form*

$$u(x,t) = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int\limits_{x-at}^{x+at} \psi(y)dy. \tag{28.2}$$

Formula (28.2) *is called d'Alembert's formula.*

Proof. The characteristic equation for Eq. (28.1) is (see Chap. 2)

$$(dx)^2 - a^2(dt)^2 = 0$$

which is equivalent to two equations

$$dx - adt = 0, \qquad dx + adt = 0,$$

whose general integrals have the form

$$\Psi_1(x,t) = x - at = C_1, \qquad \Psi_2(x,t) = x + at = C_2 \tag{28.3}$$

and, as shown below, play an important role in solving the problem.

Introduce new variables on the plane xOt:

$$\alpha = x + at, \qquad \beta = x - at. \tag{28.4}$$

Then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}, \qquad \frac{\partial u}{\partial t} = a\frac{\partial u}{\partial \alpha} - a\frac{\partial u}{\partial \beta}$$

and, correspondingly,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \alpha^2} + 2\frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 u}{\partial \beta^2}, \quad \frac{\partial^2 u}{\partial t^2} = a^2\Big(\frac{\partial^2 u}{\partial \alpha^2} - 2\frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 u}{\partial \beta^2}\Big).$$

Substitute these expressions into Eq. (28.1) and rearrange this equation to get

$$-4a^2\frac{\partial^2 u}{\partial \alpha \partial \beta} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial \alpha}\Big(\frac{\partial u}{\partial \beta}\Big) = 0. \tag{28.5}$$

Hence,

$$\frac{\partial u}{\partial \beta} = \chi(\beta), \qquad u(\alpha,\beta) = \int \chi(\beta)d\beta + f(\alpha).$$

Thus,

$$u(\alpha,\beta) = f(\alpha) + g(\beta),$$

where $f(\alpha)$ and $g(\beta)$ are arbitrary twice differentiable functions. Returning to the original variables, we obtain

$$u(x,t) = f(x+at) + g(x-at). \tag{28.6}$$

Define $f(x)$ and $g(x)$ so that the initial conditions be fulfilled. From the initial conditions (28.1), we find

$$f(x) + g(x) = \varphi(x), \qquad a[f'(x) - g'(x)] = \psi(x)$$

and

$$f(x) - g(x) = \frac{1}{a}\int\limits_{x_0}^{x} \psi(y)dy + C,$$

where C is an arbitrary constant. Hence,

$$\begin{aligned} f(x) &= \frac{1}{2}\varphi(x) + \frac{1}{2a}\int\limits_{x_0}^{x} \psi(y)dy + \frac{C}{2}, \\ g(x) &= \frac{1}{2}\varphi(x) - \frac{1}{2a}\int\limits_{x_0}^{x} \psi(y)dy - \frac{C}{2}, \end{aligned} \tag{28.7}$$

Substituting (28.7) into (28.6), we obtain

$$u(x,t) = \frac{1}{2}\varphi(x+at) + \frac{1}{2a}\int\limits_{x_0}^{x+at} \psi(y)dy + \frac{1}{2}\varphi(x-at) - \frac{1}{2a}\int\limits_{x_0}^{x-at} \psi(y)dy,$$

from which follows formula (28.2).

◊ From d'Alembert's formula it follows that the solution of the problem (28.1) is unique.

◊ The function $u(x,t)$, defined by d'Alembert's formula (28.2), describes the propagation of the initial displacements and the influence of the initial velocities of the string points on this process. Indeed, the characteristics (28.3), as follows from (28.4) and (28.6), are level curves for the functions $g(x-at)$ and $f(x+at)$. Along these curves, the functions $g(C_1)$ and $f(C_2)$ are constant. Graphically, this means that the string displacement from the position of equilibrium, fixed at a point x and equal to $g(C_1)$, uniformly propagates along the Ox-axis with velocity a in the positive direction (to the right), keeping its value. Correspondingly, the displacement $f(C_2)$ uniformly propagates along the Ox-axis with the same velocity a in the negative direction (to the left). In physics, the waves described by functions of this type are called traveling. The function $u(x,0)$ describes the string profile at $t=0$.

Thus, the general solution of the Cauchy problem for an infinite string is a superposition of two waves $f(x+at)$ (forward wave) and $g(x-at)$ (backward wave) propagating, respectively, to the right and to the left with the same velocity a.

Example 28.1. At time zero, an infinite string has the shape shown in Fig. 36. Construct the string profiles for the points in time $t_k = lk/6a$, where $l = d - c$ and $k = 1, 2, 3, 4$, and 11. Find a region where the solution of the problem, $u(x,t)$, is zero for $t > 0$.

Solution. By the data of the problem, the function $\psi(x) \equiv 0$. Then, in accordance with (28.2),

$$u(x,t) = \frac{1}{2}[\varphi(x-at) + \varphi(x+at)] = \frac{1}{2}\varphi(x-at) + \frac{1}{2}\varphi(x+at),$$

where $\varphi(x)$ is specified graphically (see Fig. 36).

At $t = 0$, the forward wave $\frac{1}{2}\varphi(x-at)$ and the backward wave $\frac{1}{2}\varphi(x+at)$ merge into one wave with the amplitude $\frac{1}{2}\varphi(x)$. In a time $t > 0$, the forward wave will move without distortion to the right for distance at and the backward wave will move to the left for at. Adding up the plots of the displaced forward and backward waves at fixed times t_k, we obtain the string profiles at these times (Fig. 37).

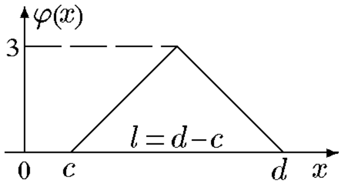

Fig. 36.

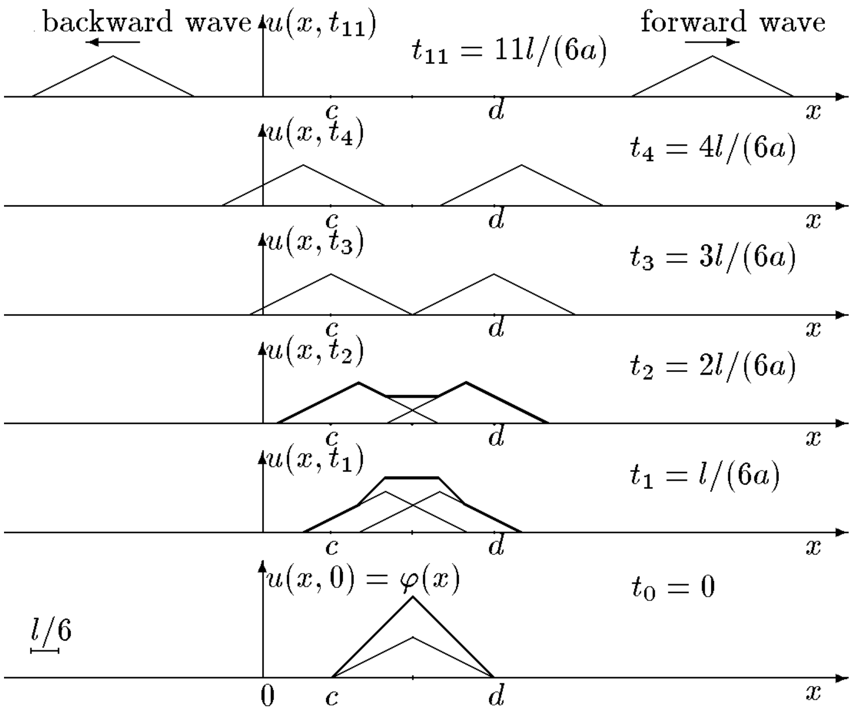

Fig. 37.

For solving the second part of the problem, we can use Fig. 37. However, since the times t_k are fixed, complete information about the regions where $u(x,t) = 0$ can be obtained by displaying the propagation of vibrations over the phase plane xOt (Fig. 38). Plot on the Ox-axis of the xOt-plane a segment $[c,d]$ corresponding to the region of initial displacement. Construct from the point d a characteristic $x - at = d$ corresponding to the displacement of the leading edge of the forward wave. Construct from the point c a characteristic $x - at = c$ describing the displacement of the trailing edge of the forward wave. The region between these characteristics is the region of propagation of the forward wave. Similarly, the characteristics $x+at = c$ and $x+at = d$ correspond to the motion of the leading and trailing edges of the backward wave and the region between them is the region of propagation of the backward wave. Thus, the above characteristics set aside three regions (I, II, and III) of the phase plane where the solution $u(x,t)$ is equal to zero.

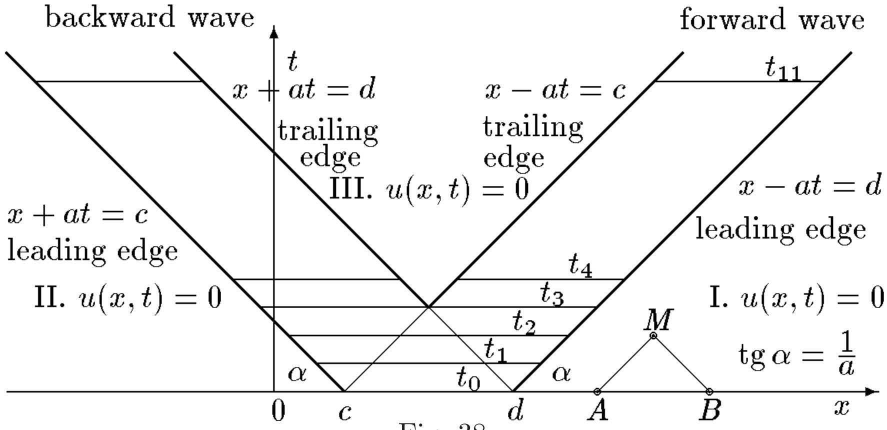

Fig. 38.

If we draw on the phase plane straight lines $t = t_k$, the points of intersection of these lines with the characteristics indicate the regions of propagation of the initial displacement from the Ox-axis. On the other hand, if we fix the coordinate x, we can follow the propagation of vibrations in time. Thus, the observer being at a point $x > d$ at time zero sees the string in equilibrium, then observes the propagation of the leading and trailing edges of the forward wave, and then again sees the string in equilibrium. The observer being at a point $x < c$ observes a similar picture. At a point $c < x < d$, the observer sees the interaction of the trailing edges of the waves after which he observes the string in equilibrium as well. In all cases, at any point x, as a certain time elapses, the string returns to the position of equilibrium. In this case, Huygens' principle is said to be valid.

Naturally, all the foregoing holds provided that the string is infinite and the boundary conditions have no effect on its behavior in the region of finite values of x. This enables one to estimate the string length from the condition $L \gg aT$, where $]0, T[$ is the time interval in which the vibrational process is considered.

In conclusion, we give a geometric interpretation of d'Alembert's formula for the solution of the problem under consideration: the solution $u(M)$ at a fixed point $M(x_M, t_M)$ of the phase plane is determined only by the initial displacements at the points $A(x_M - at_m, 0)$ and $B(x_M + at_M, 0)$ where the Ox-axis intersects with the characteristics passing through the point M (Fig. 38). The triangle MAB is called the basic triangle for the point M. From the viewpoint of the observer being at the vertex M of the characteristic triangle, the string can be considered infinite if the point M is at a distance over at_M from the string ends. The points A and B are points where the initial value of $\varphi(x)$ affects the value of the solution $u(x, t)$ at the point M.

It should also be noted that the plots of the function $u(x, t)$ in these examples contain piecewise lines and discontinuities, which contradicts the requirement of continuous differentiability of the function $u(x, t)$ and its partial derivatives, the second ones included. This contradiction can be treated in two ways. On the one hand, it can be assumed that the functions are smooth in the neighborhood of discontinuities and the piecewise lines are used just for convenience. On the other hand, we can consider the solution of the problem (28.1) in the class of generalized functions which admit such solutions and, moreover, in other physical interpretations (acoustic waves, etc.) having natural physical substantiations (shock waves).

29 Initial boundary value problems for the one-dimensional nonhomogeneous wave equation

29.1 The Cauchy problem

Let us consider a vibrating infinite string subject to the action of an external force characterized by a quantity $F(x,t)$, a force related to a unit mass:

$$\begin{gathered} u_{tt} - a^2 u_{xx} = F(x,t), \qquad -\infty < x < \infty, \quad t > 0; \\ u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x). \end{gathered} \tag{29.1}$$

Equation (29.1) is a nonhomogeneous wave equation.

As for the homogeneous equation (28.1), we use relations (28.4) to introduce new variables η and ξ. Equation (29.1) will then take the form (28.5)

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = -\frac{1}{4a^2}\bar{F}(\eta,\xi), \tag{29.2}$$

where $\bar{F}(\eta,\xi)$ is the function $F(x,y)$ written in the variables η and ξ.

While the solution of the problem (29.1) in the variables x,t is sought in the region $0 < x < \infty$, $t > 0$ (the top semiplane xOt) with initial conditions specified on the straight line $t = 0$, the solution of this problem in the variables η,ξ is sought in the region $-\infty < \eta+\xi < \infty$ ($x = (\eta+\xi)/2$), $\xi - \eta > 0$ ($t = (\xi-\eta)/2a$) with "initial conditions" specified on the straight line $\xi - \eta = 0$ (Fig. 39).

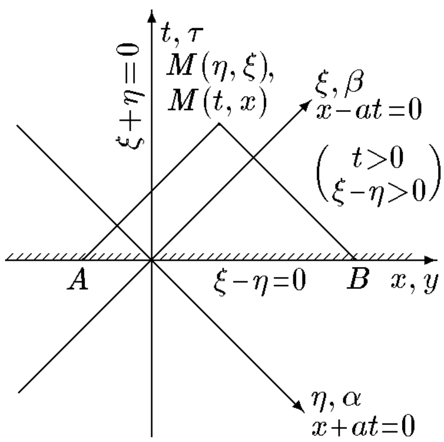

Fig. 39.

To find the general solution, we rewrite the second order partial differential equation (29.2) as a system of two first order equations [see also (8.7)]:

$$\frac{\partial Z(\eta,\xi)}{\partial\eta} = -\frac{1}{4a^2}\bar{F}(\eta,\xi), \tag{29.3}$$

$$\frac{\partial u(\eta,\xi)}{\partial\xi} = Z(\eta,\xi). \tag{29.4}$$

Integration of Eq. (29.3) with respect to η on condition that $\xi = \text{const}$ yields

$$Z(\eta,\xi) = \int\limits_{\beta}^{\eta}\Big(-\frac{1}{4a^2}\Big)\bar{F}(\alpha,\xi)d\alpha + p(\xi), \tag{29.5}$$

and integration of (29.4) with respect to ξ on condition that $\eta = \text{const}$, correspondingly, gives

$$u(\eta,\xi) = \int\limits_{\eta}^{\xi} Z(\eta,\beta)d\beta + q(\eta), \tag{29.6}$$

where $p(\xi)$ and $q(\eta)$ are arbitrary functions of the variables ξ and η.

Substitution of (29.5) into (29.6) yields the general solution in the form

$$u(\eta,\xi)=\int\limits_{\eta}^{\xi}\Big[-\frac{1}{4a^2}\int\limits_{\beta}^{\eta}\bar F(\alpha,\beta)d\alpha+p(\beta)\Big]d\beta+q(\eta)$$

or

$$u(\eta,\xi)=-\frac{1}{4a^2}\int\limits_{\eta}^{\xi}\Big[\int\limits_{\beta}^{\eta}\bar F(\alpha,\beta)d\alpha\Big]d\beta+\int\limits_{\eta}^{\xi}p(\beta)d\beta+q(\eta). \tag{29.7}$$

The last two terms in (29.7) can be represented by the sum of two new arbitrary functions $g(\eta)$ and $f(\xi)$. If we change the order of integration in the integral (29.7) and consider the solution as a solution at point $M(\eta,\xi)$, the general solution (29.7) becomes

$$u(\eta,\xi)=\frac{1}{4a^2}\int\limits_{\xi}^{\eta}\Big[\int\limits_{\beta}^{\eta}\bar F(\alpha,\beta)d\alpha\Big]d\beta+g(\eta)+f(\xi), \tag{29.8}$$

where the integration is performed over the region

$$\eta<\alpha<\beta<\xi \tag{29.9}$$

that corresponds to the rectangle MAB in Fig. 39. In view of this, the solution (29.8) can be represented by a double integral of the form

$$u(\eta,\xi)=\frac{1}{4a^2}\iint\limits_{\triangle MAB}\bar F(\alpha,\beta)d\alpha\,d\beta+g(\eta)+f(\xi). \tag{29.10}$$

Returning to the original variables, in view of

$$\frac{D(\eta,\xi)}{D(t,x)}=\frac{D(\alpha,\beta)}{D(\tau,y)}=\begin{vmatrix}1 & -a\\ 1 & a\end{vmatrix}=2a$$

and $\bar F(\alpha,\beta)=F(y,\tau)$, we have

$$u(x,t)=\frac{1}{2a}\iint\limits_{\triangle MAB}F(y,\tau)d\tau\,dy+g(x-at)+f(x+at). \tag{29.11}$$

To calculate the double integral, we write the equations of the straight lines MA and MB in the coordinates τ,y as equations of straight lines with angular coefficients $\pm1/a$, passing through a given point $M(t,x)$:

$$\tau-t=\frac{1}{a}(y-x),$$
$$\tau-t=-\frac{1}{a}(y-x),$$

whence

$$\begin{aligned}y&=x-a(t-\tau),\\ y&=x+a(t-\tau).\end{aligned} \tag{29.12}$$

Write the double integral in formula (29.11) as an iterated one where the external integral is calculated with respect to the variable τ. Then

$$u(x,t) = \frac{1}{2a}\int\limits_0^t d\tau \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} F(y,\tau)dy + g(x-at) + f(x+at). \tag{29.13}$$

Find the arbitrary functions g and f from the initial conditions (29.1). Differentiate (29.13) with respect to t in view of the rule of differentiation of definite integrals with a variable upper limit:

$$\frac{d}{dt}\int\limits_{\vartheta(t)}^{\gamma(t)} f(y,t)dy = \int\limits_{\vartheta(t)}^{\gamma(t)} \frac{\partial f}{\partial t}(y,t)dy + \gamma'(t)f(\gamma(t),t) - \vartheta'(t)f(\vartheta(t),t). \tag{29.14}$$

Then

$$u_t(x,t) = \int\limits_0^t \frac{1}{2}\big[F(x+a(t-\tau),\tau) + F(x-a(t-\tau),\tau)\big]d\tau -$$
$$-ag'(x-at) + af'(x+at). \tag{29.15}$$

Substitution of (29.13) and (29.15) into the initial conditions (29.1) yields the system of equations

$$g(x) + f(x) = \varphi(x),$$
$$a[f'(x) - g'(x)] = \psi(x)$$

which completely coincides with that obtained in solving the homogeneous problem (28.1) and gives a solution described by d'Alembert's formula (28.2):

$$g(x-at) + f(x+at) = \frac{1}{2}\big[\varphi(x-at) + \varphi(x+at)\big] + \frac{1}{2}\int\limits_{x-at}^{x+at} \psi(y)dy.$$

Thus, the final solution of the nonhomogeneous problem (29.1) can be written

$$u(x,t) = \frac{1}{2a}\int\limits_0^t d\tau \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} F(y,\tau)dy + \frac{1}{2a}\int\limits_{x-at}^{x+at} \psi(y)dy + \frac{1}{2}\big[\varphi(x+at) + \varphi(x-at)\big]. \tag{29.16}$$

From formula (29.16) it is obvious that the solution of the Cauchy problem for the nonhomogeneous wave equation with homogeneous initial conditions, i.e., the problem

$$\begin{aligned} &\tilde{u}_{tt} - a^2\tilde{u}_{xx} = F(x,t), \\ &\tilde{u}(x,0) = \tilde{u}_t(x,0) = 0, \end{aligned} \tag{29.17}$$

is given by the function

$$\tilde{u}(x,t) = \frac{1}{2a}\int\limits_0^t d\tau \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} F(y,\tau)dy. \tag{29.18}$$

Let us show that this formula can be obtained from the solution of the Cauchy problem for the homogeneous equation (28.1) with a rule of thumb, which is also used below and called Duhamel's principle.

Statement 29.1 (Duhamel's principle). *The solution of the problem* (29.17) *can be represented in the form*

$$u(x,t) = \int\limits_0^t v(x,t,\tau)d\tau, \tag{29.19}$$

where the function $v(x,t,\tau)$ *is the solution of the following problem:*

$$\begin{gathered} v_{tt} - a^2 v_{xx} = 0, \\ v\Big|_{t=\tau} = 0, \quad v_t\Big|_{t=\tau} = F(x,\tau). \end{gathered} \tag{29.20}$$

Indeed, according to d'Alembert's formula (28.2), the solution of the problem (29.20) has the form

$$v(x,t,\tau) = \frac{1}{2a} \int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} F(y,\tau)dy. \tag{29.21}$$

Substitution of (29.21) into (29.19) yields (29.18), which proves the statement. Moreover, the truth of the statement can be proved by direct check.

To do this, write the derivatives of the function $u(x,t)$, (29.18), in view of relation (29.14), in the form

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{a}{2a}\int\limits_0^t d\tau\,[F(x+a(t-\tau),\tau) + F(x-a(t-\tau),\tau)];\\
\frac{\partial^2 u}{\partial t^2} &= F(x,t) + \frac{a}{2}\int\limits_0^t d\tau\,\frac{\partial}{\partial x}[F(x+a(t-\tau),\tau) - F(x-a(t-\tau),\tau)];\\
\frac{\partial u}{\partial x} &= \frac{1}{2a}\int\limits_0^t d\tau\,[F(x+a(t-\tau),\tau) - F(x-a(t-\tau),\tau)];\\
\frac{\partial^2 u}{\partial x^2} &= \frac{1}{2a}\frac{\partial}{\partial x}\int\limits_0^t d\tau\,[F(x+a(t-\tau),\tau) - F(x-a(t-\tau),\tau)].
\end{aligned}$$

Substituting $\partial^2 u/\partial t^2$ and $\partial^2 u/\partial x^2$ into (29.17), we arrive at the identity $0 = 0$, that is $u(x,t)$ (29.18) is solution of the nonhomogeneous wave equations.

30 The Fourier method and mixed problems for the one-dimensional wave equation on a finite interval

Let us consider the problem of transverse vibrations of a string. The mathematical statement of this problem is (see Sec. "Equations of a vibrating string")

$$u_{tt} - a^2 u_{xx} = f(x,t), \quad 0 \leqslant x \leqslant l, \quad t > 0, \tag{30.1}$$
$$u|_{t=0} = \varphi(x), \qquad u_t|_{t=0} = \psi(x), \tag{30.2}$$
$$(\alpha_1 u + \beta_1 u_x)|_{x=0} = \mu_1(t), \qquad (\alpha_2 u + \beta_2 u_x)|_{x=l} = \mu_2(t),$$

where $a^2 = T/\rho$. Let us first consider a mixed problem with boundary conditions of the first kind, i.e., $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 0$. Analyze the problem in more detail and using it as an example, follow the relation between the Fourier method and the Green's function method.

30.1 The homogeneous mixed problem with homogeneous boundary conditions of the first kind

Consider a vibrating string of length l with fixed ends. This physical process is associated mathematically with the following mixed problem:

$$u_{tt} - a^2 u_{xx} = 0, \quad 0 \leqslant x \leqslant l, \quad t > 0, \tag{30.3}$$
$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \tag{30.4}$$
$$u(0,t) = u(l,t) = 0.$$

We seek a solution by the method of separation of variables. For doing this, represent the function $u(x,t)$ in the form

$$u(x,t) = X(x)T(t). \tag{30.5}$$

Then

$$\frac{T''}{T} - a^2\frac{X''}{X} = 0.$$

Denote

$$\frac{X''}{X} = \lambda, \qquad \frac{T''}{T} = a^2\lambda.$$

For the function $X(x)$, we arrive at the Sturm–Liouville problem

$$\begin{aligned} &X'' - \lambda X = 0, \\ &X(0) = X(l) = 0, \quad \text{otherwise} \quad T(t) \equiv 0. \end{aligned} \tag{30.6}$$

The solution of the Sturm–Liouville problem (30.6) was obtained in Example III.2.2:

$$X_n(x) = A_n \sin\frac{\pi n x}{l}, \qquad \lambda_n = -\Big(\frac{\pi n}{l}\Big)^2, \quad n = \overline{0,\infty}. \tag{30.7}$$

Substitute λ_n (30.7) into the equation for $T(t)$ to get

$$T'' + \Big(\frac{\pi n}{l}\Big)^2 a^2 T = 0.$$

The general solution of this equation has the form

$$T_n(t) = C_n \cos\frac{\pi n a}{l}t + D_n \sin\frac{\pi n a}{l}t. \tag{30.8}$$

Taking into account that $u_n(x,t) = X_n(x)T_n(t)$ and summing up over all n, we obtain

$$u(x,t) = \sum_{n=1}^{\infty} \Big[\bar{C}_n \cos\frac{\pi ant}{l} + \bar{D}_n \sin\frac{\pi ant}{l}\Big] \sin\frac{\pi nx}{l}, \tag{30.9}$$

where $\bar{C}_n = A_n C_n$ and $\bar{D}_n = A_n D_n$. In this case, the initial conditions

$$u(x,0) = \sum_{n=1}^{\infty} \bar{C}_n \sin\frac{\pi nx}{l} = \varphi(x) = \sum_{n=1}^{\infty} \alpha_n \sin\frac{\pi n}{l}x;$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \bar{D}_n \frac{\pi na}{l} \sin\frac{\pi nx}{l} = \psi(x) = \sum_{n=1}^{\infty} \beta_n \sin\frac{\pi n}{l}x$$

must be fulfilled. Here α_n and β_n are the Fourier coefficients of the functions $\varphi(x)$ and $\psi(x)$, respectively:

$$\begin{aligned} \alpha_n &= \frac{2}{l}\int\limits_0^l \varphi(y)\sin\frac{\pi ny}{l}dy; \\ \beta_n &= \frac{2}{l}\int\limits_0^l \psi(y)\sin\frac{\pi ny}{l}dy. \end{aligned} \tag{30.10}$$

Two Fourier series of orthogonal functions are identical if the corresponding coefficients of these series are equal. Hence,

$$\bar{C}_n = \alpha_n, \qquad \bar{D}_n = \frac{l\beta_n}{\pi an}.$$

As a result, we obtain

$$u(x,t) = \sum_{n=1}^{\infty} \Big[\alpha_n \cos\frac{\pi ant}{l} + \frac{l\beta_n}{\pi an} \sin\frac{\pi ant}{l}\Big] \sin\frac{\pi nx}{l}, \tag{30.11}$$

Example 30.1. Find the law of the vibrations of a string of length 1/2 with fixed ends. The initial velocity is equal to zero and the initial displacement is given by the function $u(x,0) = x(x - 1/2)$.

Solution. This physical process is associated mathematically with the mixed problem

$$u_{tt} = a^2 u_{xx}; \tag{30.12}$$

$$u(x,0) = x(x-1/2),\ u_t(x,0) = u(0,t) = u(1/2,t) = 0 \tag{30.13}$$

for the half-strip $0 < x < 1/2$, $0 < t < \infty$. Seek a solution by the method of separation of variable. To do this, represent the function $u(x,t)$ in the form

$$u(x,t) = X(t)T(t). \tag{30.14}$$

Denote

$$\frac{dX}{dx} = X', \qquad \frac{dT}{dt} = \dot{T}.$$

Substitute (30.12) into (30.14) to get

$$\ddot{T}X = a^2X''T, \qquad \frac{\ddot{T}}{T} = a^2\frac{X''}{X} = \lambda$$

or

$$\ddot{T} - \lambda T = 0, \qquad X'' - \frac{1}{a^2}\lambda X = 0.$$

Substitute the function (30.12) into the boundary conditions (30.13)

$$X(0)T(t) = X(1/2)T(t) = 0.$$

Thus,

$$X(0) = 0, \qquad X(1/2) = 0.$$

Denote $\lambda = \tilde{\lambda}a^2$, and then

$$\dot{T} - \tilde{\lambda}a^2T, \qquad X'' - \tilde{\lambda}X = 0.$$

For $X(x)$, we arrive at the Sturm–Liouville problem

$$X'' - \tilde{\lambda}X = 0, \qquad X(0) = X(1/2) = 0,$$

whose solution has the form (see Example III.2.1)

$$X_n(x) = B_n \sin 2\pi nx.$$

For the function $T(t)$, we obtain

$$\ddot{T} + 4\pi^2n^2a^2T = 0.$$

Compose a characteristic equation

$$k^2 + 4\pi^2n^2a^2 = 0$$

whose roots are

$$k = \pm i2\pi na.$$

Hence,

$$T_n(t) = C_n \cos 2\pi nat + D_n \sin 2\pi nat$$

and

$$u_n(x,t) = X_n(x)T_n(t).$$

Summation over n yields a solution of Eq. (30.12) satisfying the boundary conditions (30.13):

$$u(x,t) = \sum_{n=1}^{\infty} \big[\widetilde{C}_n \cos 2\pi nat + \widetilde{D}_n \sin 2\pi nat\big] \sin 2\pi nx.$$

Here $\widetilde{D}_n = D_nB_n$ and $\widetilde{C}_n = C_nB_n$ are constants to be determined from the boundary conditions (30.13). From the first condition we find

$$u(x,0) = \sum_{n=1}^{\infty} \widetilde{C}_n \sin 2\pi nx = x\Big(x - \frac{1}{2}\Big) = \sum_{n=1}^{\infty} \alpha_n \sin 2\pi nx$$

with the Fourier coefficients of the function $\varphi(x) = x(x - 1/2)$ given by

$$\alpha_n = \frac{2}{l}\int\limits_0^l \varphi(x)\sin(2\pi nx/l)dx.$$

Equating the coefficients of identical functions $\sin 2\pi nx$, we obtain

$$\widetilde{C}_n = \alpha_n.$$

Similarly, from the second boundary condition

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin 2\pi nx(-\widetilde{C}_n 2\pi na \sin 2\pi nat + \widetilde{D}_n 2\pi na \cos 2\pi nat),$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \sin 2\pi nx \widetilde{D}_n 2\pi na = 0$$

it follows that

$$\widetilde{D}_n = 0.$$

Find the explicit form of the coefficients α_n:

$$\alpha_n = 4\int_0^{1/2} x(x-1/2)\sin(2\pi nx)\,dx.$$

Integrate by parts, putting

$$U = x(x-1/2), \qquad dU = (2x-1/2)dx;$$
$$dV = \sin 2\pi nx\,dx, \qquad V = -\frac{1}{2\pi n}\cos 2\pi nx,$$

to get

$$\alpha_n = 4\Big[x(x-1/2)\Big(-\frac{\cos 2\pi nx}{2\pi n}\Big)\Big|_0^{1/2} - \frac{1}{2\pi n}\int_0^{1/2}(2x-1/2)\cos 2\pi nx\,dx\Big].$$

Integrate by parts once more:

$$\alpha_n = -\frac{4}{2\pi n}\Big(-\frac{1}{\pi n}\Big)\Big(-\frac{\cos 2\pi nx}{2\pi n}\Big)\Big|_0^{1/2} = -\frac{1}{\pi^3 n^3}[-(-1)^n+1] = \frac{1}{\pi^3 n^3}[(-1)^n-1],$$

i.e.,

$$\alpha_{2k} = 0, \qquad \alpha_{2k+1} = -\frac{2}{\pi^3(2k+1)^3}.$$

Hence,

$$\widetilde{C}_n = \frac{1}{\pi^3 n^3}[(-1)^n - 1] = \begin{cases} 0, & n = 2k; \\ -\dfrac{2}{\pi^3(2k+1)^3}, & n = 2k+1, \end{cases}$$

where $k = \overline{0,\infty}$. Finally, we obtain

$$u(x,t) = -\frac{2}{\pi^3}\sum_{k=0}^{\infty}\frac{\sin[2\pi(2k+1)x]\cos[2\pi(2k+1)at]}{(2k+1)^3}.$$

In view of the relation

$$\sin\alpha\cos\beta = \frac{1}{2}\big[\sin(\alpha+\beta) + \sin(\alpha-\beta)\big],$$

we write

$$u(x,t) = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)],$$
$$\varphi(\xi) = -\frac{2}{\pi^3}\sum_{n=0}^{\infty}\frac{\sin 2\pi(2n+1)\xi}{(2n+1)^3}.$$

The function $\varphi(\xi)$ has the properties

$$\varphi(\xi) = -\varphi(-\xi), \quad \varphi(\xi+1/2) = -\varphi(\xi), \quad \varphi(\xi+1) = \varphi(\xi)$$

and, on the interval $[0,1/2]$, has the form

$$\varphi(\xi) = \xi(\xi - 1/2).$$

From here it follows that

$$\varphi(\xi) = \begin{cases} (\xi - [\xi])\Big(\xi - [\xi] - \dfrac{1}{2}\Big), & 0 \leqslant \xi - [\xi] \leqslant 1/2; \\ (1 - \xi - [\xi])\Big(\xi - [\xi] - \dfrac{1}{2}\Big), & \xi - [\xi] > 1/2, \end{cases}$$

where $[\xi]$ is the integer part of ξ.

30.2 The fundamental solution of a mixed problem with boundary conditions of the first kind

♦ By the Green's function $g(x,y,t)$ (fundamental solution) of a mixed problem for d'Alembert's equation is meant a generalized function satisfying the conditions

$$\begin{gathered} g_{tt} = a^2 g_{xx}, \qquad t > 0, \quad 0 \leqslant x \leqslant l, \\ (\alpha_1 g_x + \beta_1 g)|_{x=0} = (\alpha_2 g_x + \beta_2 g)|_{x=l} = 0, \\ g|_{t=0} = 0, \qquad g_t|_{t=0} = \delta(x-y). \end{gathered} \tag{30.15}$$

Statement 30.1. *The Green's function of the mixed problem*

$$\begin{gathered} g_{tt} = a^2 g_{xx}, \qquad t > 0, \quad 0 < x < l, \\ g|_{x=0} = g|_{x=l} = 0, \\ g|_{t=0} = 0, \qquad g_t|_{t=0} = \delta(x-y) \end{gathered} \tag{30.16}$$

can be represented in the form

$$g(x,y,t) = \sum_{n=1}^{\infty}\frac{2}{\pi a n}\sin\frac{\pi n x}{l}\sin\frac{\pi n y}{l}\sin\frac{\pi n a t}{l}. \tag{30.17}$$

The first three conditions in (30.16) are fulfilled by virtue of (30.17). The last condition of (30.16) follows from the relation

$$\delta(x-y) = \sum_{n=1}^{\infty}\frac{2}{l}\sin\frac{\pi n x}{l}\sin\frac{\pi n y}{l}, \tag{30.18}$$

which is obtained from the expansion of the δ-function in a Fourier series of a complete system orthogonal functions [see (III.2.9)]. Thus, the statement is proved.

Let us consider the simplest properties of the Green's function of the mixed problem (30.17).

Property 1. The function

$$u(x,t) = \int_0^l \{g_t(x,y,t)\varphi(y) + g(x,y,t)\psi(y)\}dy \tag{30.19}$$

is the solution of the problem (30.3), (30.4).

Proof. Substituting (30.10) into (30.11) and interchanging the summation and the integration, we obtain (30.19), where $g(x,y,t)$ is defined in (30.17).

Property 2. The function $\mathfrak{g}(x,y,t) = g_t(x,y,t)$ is a solution of the mixed problem

$$\begin{gathered} \mathfrak{g}_{tt} = a^2\mathfrak{g}_{xx}, \qquad t>0, \quad 0 \leqslant x \leqslant l, \\ \mathfrak{g}|_{x=0} = \mathfrak{g}|_{x=l} = 0, \\ \mathfrak{g}|_{t=0} = \delta(x-y), \quad \mathfrak{g}_t|_{t=0} = 0. \end{gathered} \tag{30.20}$$

Proof is completely similar to that of the preceding statement.

Property 3. The function

$$\mathfrak{E}(x,y,t) = \theta(t)g(x,y,t) \tag{30.21}$$

is the fundamental solution of the mixed problem (30.3), (30.4), i.e.,

$$\begin{gathered} \mathfrak{E}_{tt} - a^2\mathfrak{E}_{xx} = \delta(x-y)\delta(t), \\ \mathfrak{E}|_{x=0} = \mathfrak{E}|_{x=l} = \mathfrak{E}|_{t=0} = \mathfrak{E}_t|_{t=0} = 0. \end{gathered} \tag{30.22}$$

Proof. Substitute (30.21) into the left side of relation (30.22) to get

$$\mathfrak{E}_{tt} - a^2\mathfrak{E}_{xx} = \theta''_{tt}(t)g + 2\theta'_t(t)g_t + \theta(t)g_{tt} - a^2\theta(t)g_{xx} = \delta'(t)g + 2\delta(t)g_t.$$

Here we have used the relation $\theta'(t) = \delta(t)$ and Eq. (30.16).

Now we make use of the properties of the Dirac δ-function (see Sec. "The Dirac delta-function and its properties" of Part II)

$$\begin{gathered} g_t\delta(t) = g_t|_{t=0}\delta(t), \\ g\delta'(t) = g|_{t=0}\delta'(t) - g_t|_{t=0}\delta(t). \end{gathered}$$

Then

$$\mathcal{E}_{tt} - a^2\mathcal{E}_{xx} = g|_{t=0}\delta'(t) + g_t|_{t=0}\delta(t) = \delta(x-y)\delta(t).$$

Here we have applied the initial conditions (30.16) to the function $g(x,y,t)$. The initial and boundary conditions (30.22) follow from the corresponding conditions for the function $g(x,y,t)$. Thus, Property 3 is proved.

Property 4. The function

$$\tilde{u}(x,t) = \int_0^t d\tau \int_0^l g(x,y,t-\tau)f(y,\tau)dy \tag{30.23}$$

is a solution of the problem

$$\begin{gathered} \tilde{u}_{tt} - a^2\tilde{u}_{xx} = f(x,t), \\ \tilde{u}|_{x=0} = \tilde{u}|_{x=l} = \tilde{u}|_{t=0} = \tilde{u}_t|_{t=0} = 0. \end{gathered} \tag{30.24}$$

Proof. In view of relation (30.23), we have

$$\tilde{u}_{xx}(x,t) = \int_0^t d\tau \int_0^l g_{xx}(x,y,t-\tau)f(y,\tau)dy$$

and, similarly,

$$\tilde{u}_t = \int_0^t d\tau \int_0^l g_t(x,y,t-\tau)f(y,\tau)dy + \int_0^l g(x,y,0)f(y,t)dy. \tag{30.25}$$

By virtue of the initial conditions (30.16) for the function $g(x,y,t)$, the last term is equal to zero. Thus, the validity of the initial boundary conditions in (30.25) is obvious. Differentiation of relation (30.25) with respect to t yields

$$\tilde{u}_{tt} = \int_0^t d\tau \int_0^l g_{tt}(x,y,t-\tau)f(y,\tau)dy + \int_0^l g_t(x,y,0)f(y,t)dy.$$

Thus, in view of this relation and the definition of Green's function $g(x,y,t)$ in (30.16) and by virtue of the properties of the δ-function, Property 4 is proved.

30.3 The nonhomogeneous mixed problem with nonhomogeneous boundary conditions of the first kind

We seek nonzero solutions of the equation

$$u_{tt} = a^2 u_{xx} + f(x,t), \tag{30.26}$$

satisfying the conditions

$$u(0,t) = \mu_1(t), \qquad u(l,t) = \mu_2(t), \tag{30.27}$$

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x). \tag{30.28}$$

The solution of the problem can be carried out in several steps.

1. Introduce a new unknown function $v(x,t)$, putting

$$u(x,t) = w(x,t) + v(x,t),$$

where $w(x,t)$ is an arbitrary function satisfying the conditions $w(0,t) = \mu_1(t)$ and $w(l,t) = \mu_2(t)$. The simplest form of such a function is

$$w(x,t) = \mu_1(t) + \frac{x}{l}[\mu_2(t) - \mu_1(t)].$$

Then, for the function $v(x,t)$, we arrive at a mixed problem with nonzero (homogeneous) boundary conditions:

$$v_{tt} - a^2 v_{xx} = \bar{f}(x,t), \tag{30.29}$$

$$v(0,t) = v(l,t) = 0, \quad v(x,0) = \bar{\varphi}(x), \quad v_t(x,0) = \bar{\psi}(t),$$

where

$$\bar{f}(x,t) = f(x,t) - [w_{tt} - a^2 w_{xx}],$$
$$\bar{\varphi}(x) = \varphi(x) - w(x,0),$$
$$\bar{\psi}(t) = \psi(x) - w_t(x,0).$$

2. The solution of the mixed problem (30.29) can be represented in the form

$$v(x,t) = L(x,t) + M(x,t), \tag{30.30}$$

where

$$\begin{gathered} L_{tt} - a^2 L_{xx} = 0, \\ L|_{t=0} = \bar{\varphi}(x), \quad L_t|_{t=0} = \bar{\psi}(x), \quad L|_{x=0} = L|_{x=l} = 0 \end{gathered} \tag{30.31}$$

and

$$\begin{gathered} M_{tt} - a^2 M_{xx} = \bar{f}(x,t), \\ M|_{t=0} = M_t|_{t=0} = M|_{x=0} = M|_{x=l} = 0. \end{gathered} \tag{30.32}$$

3. According to (30.11), the solution of the mixed problem (30.31) has the form

$$L(x,t) = \sum_{n=1}^{\infty} \Big[\bar{\alpha}_n \cos\frac{\pi a n t}{l} + \frac{\bar{\beta}_n l}{\pi a n}\sin\frac{\pi a n t}{l}\Big]\sin\frac{\pi n x}{l}, \tag{30.33}$$

where

$$\bar{\alpha}_n = \frac{2}{l}\int\limits_0^l \bar{\varphi}(y)\sin\frac{\pi n y}{l}dy, \quad \bar{\beta}_n = \frac{2}{l}\int\limits_0^l \bar{\psi}(y)\sin\frac{\pi n y}{l}dy.$$

4. The solution of the problem (30.32) is sought as an expansion in the eigenfunctions of the Sturm–Liouville problem (30.6)

$$M(x,t) = \sum_{n=1}^{\infty} \Theta_n(t)\sin\frac{\pi n x}{l}, \tag{30.34}$$

where the functions $\Theta_n(t)$ are to be determined. From (30.32) it follows that

$$\Theta_n(0) = \Theta_n'(0) = 0. \tag{30.35}$$

Substitution of (30.34) into (30.32) yields

$$\sum_{n=1}^{\infty} \Big[\Theta_n'' + a^2\Big(\frac{\pi n}{l}\Big)^2\Theta_n\Big]\sin\frac{\pi n x}{l} = \bar{f}(x,t).$$

Expand the right side of this equation into a Fourier series in the eigenfunctions of the Sturm–Liouville problem (30.6)

$$\bar{f}(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin\frac{\pi n x}{l}, \tag{30.36}$$

$$f_n(t) = \frac{2}{l}\int\limits_0^l \bar{f}(y,t)\sin\frac{\pi n y}{l}dy$$

and equate the coefficients of the Fourier series on the right and on the left side. Then, for the function $\Theta_n(t)$ we obtain the following equation:

$$\Theta_n'' + \Big(\frac{\pi n a}{l}\Big)^2\Theta_n = f_n(t) \tag{30.37}$$

with the initial conditions (30.35).

We seek the solution of Eq. (30.37) by the Lagrange method:

$$\Theta_n(t) = p_n(t)\cos\omega_n t + q_n(t)\sin\omega_n t, \qquad \omega_n = \frac{\pi a n}{l},$$

where the functions $p_n(t)$ and $q_n(t)$ are determined from the system of equations

$$\begin{cases} p_n'(t)\cos\omega_n t + q_n'(t)\sin\omega_n t = 0, \\ p_n'(t)(-\omega_n)\sin\omega_n t + q_n'(t)\omega_n\cos\omega_n t = f_n(t). \end{cases}$$

As a result, we have

$$p_n(t) = -\frac{1}{\omega_n}\int\limits_0^t f_n(\tau)\sin\omega_n\tau d\tau + p_n^0,$$

$$q_n(t) = \frac{1}{\omega_n}\int\limits_0^t f_n(\tau)\cos\omega_n\tau d\tau + q_n^0.$$

From the initial conditions (30.35), $q_n^0 = p_n^0 = 0$. Hence,

$$\Theta_n(t) = \frac{1}{\omega_n}\int\limits_0^t f_n(\tau)(-\sin\omega_n\tau\cos\omega_n t + \cos\omega_n\tau\sin\omega_n t)d\tau =$$

$$= \frac{1}{\omega_n}\int\limits_0^t f_n(\tau)\sin\omega_n(t-\tau)d\tau.$$

Thus,

$$M(x,t) = \sum_{n=1}^{\infty}\frac{1}{\omega_n}\int\limits_0^t f_n(\tau)\sin\omega_n(t-\tau)d\tau\sin\frac{\omega_n x}{a}. \tag{30.38}$$

Substitution of (30.36) into (30.38) yields

$$M(x,t) = \int\limits_0^t d\tau\int\limits_0^l f(y,\tau)g(x,y,t-\tau)dy, \tag{30.39}$$

where the function $g(x,y,t-\tau)$ is defined by relation (30.17).

Finally,

$$u(x,t) = v(x,t) + w(x,t); \tag{30.40}$$
$$w(x,t) = \Big(1 - \frac{x}{l}\Big)\mu_1(t) + \frac{x}{l}\mu_2(t),$$
$$v(x,t) = L(x,t) + M(x,t),$$

where the functions $L(x,t)$ and $M(x,t)$ are given by relations (30.33) and (30.39), respectively.

Example 30.2. Solve the mixed problem

$$u_{tt} = u_{xx}, \qquad 0 < x < \pi, \qquad t > 0; \tag{30.41}$$
$$u(0,t) = t^2, \quad u(\pi,t) = t^2, \quad u(x,0) = \sin x, \quad u_t = 0.$$

Solution. We seek a solution, following to the general scheme (30.40), in the form

$$u(x,t) = v(x,t) + \Big(1 - \frac{x}{\pi}\Big)t^2 + \frac{x}{\pi}t^2 = v(x,t) + t^2.$$

For the function $v(x,t)$, we have the following mixed problem:

$$\begin{gathered} v_{tt} = v_{xx} - 2; \qquad v(0,t) = 0, \quad v(\pi,t) = 0; \\ v(x,0) = \sin x, \qquad v_t(x,0) = 0. \end{gathered} \tag{30.42}$$

Seek a solution of the problem (30.42) in the form

$$v(x,t) = L(x,t) + M(x,t),$$

where $L(x,t)$ is a solution of the homogeneous equation

$$\begin{aligned} &L_{tt} = L_{xx}; \\ &L|_{x=0} = L|_{x=\pi} = 0; \\ &L|_{t=0} = \sin x; \qquad L_t|_{t=0} = 0, \end{aligned} \tag{30.43}$$

and $M(x,t)$ is a solution of the nonhomogeneous equation

$$M_{tt} = M_{xx} + 2 \tag{30.44}$$

with nonzero boundary and initial conditions

$$M|_{x=0} = M|_{x=\pi} = 0; \qquad M|_{t=0} = M_t|_{t=0} = 0.$$

Seek a particular solution of the problem (30.43) in the form

$$L(x,t) = X(x)T(t).$$

Separating the variables, we arrive at the following equation for the function $T(t)$:

$$T'' - \lambda T = 0.$$

In view of the boundary conditions

$$L(0,t) = X(0)T(t) = 0, \qquad L(\pi,t) = X(\pi)T(t) = 0,$$

for the function $X(x)$, we obtain the Sturm–Liouville problem

$$X'' - \lambda X = 0, \qquad X(0) = X(\pi) = 0,$$

whose solution has the form

$$X_n(x) = B_n \sin nx; \qquad \lambda_n = -n^2, \quad n = \overline{1,\infty}.$$

Then, for the functions $T_n(t)$, we have

$$T_n'' + n^2 T_n = 0, \qquad n = \overline{1,\infty}.$$

From the characteristic equation $\omega^2 + n^2 = 0$ we find $\omega = \pm in$. Therefore, the general solution is

$$T_n(t) = C_n \cos nt + D_n \sin nt.$$

Hence,

$$L_n(x,t) = (\bar{C}_n \cos nt + \bar{D}_n \sin nt)\sin nx,$$

where $\bar{C}_n = C_n B_n$, $\bar{D}_n = D_n B_n$. Summing up over n, we get

$$L(x,t) = \sum_{n=1}^{\infty} (\bar{C}_n \cos nt + \bar{D}_n \sin nt)\sin nx.$$

From the initial condition for the function $u(x,t)$, we find

$$L(x,0) = \sum_{n=1}^{\infty} \bar{C}_n \sin nx = \sin x + \sum_{n=2}^{\infty} 0 \cdot \sin nx.$$

Hence,

$$\bar{C}_n = \delta_{n1}.$$

From the initial condition for the derivative $u_t(x,t)$, we find analogously

$$L_t(x,0) = \sum_{n=1}^{\infty} \sin x[-n\bar{C}_n \cdot 0 + \bar{D}_n n] = 0 = \sum_{n=1}^{\infty} 0 \cdot \sin nx.$$

Hence,

$$\bar{D}_n = 0, \qquad n = \overline{0,\infty},$$

and, finally,

$$L(x,t) = \sum_{n=1}^{\infty} \cos nt \sin x\, \delta_{1n} = \cos t \sin x. \tag{30.45}$$

Seek a solution of the problem (30.44) in the form

$$M(x,t) = \sum_{n=1}^{\infty} \Theta_n(t) \sin nx. \tag{30.46}$$

Substitution of (30.46) into (30.44) yields

$$\sum_{n=1}^{\infty} \Theta_n''(t) \sin nx = \sum_{n=1}^{\infty} n^2(-\sin nx)\Theta_n(t) + 2.$$

Expand the right side of this equation in a Fourier series in $\sin nx$:

$$2 = \sum_{n=1}^{\infty} \alpha_n \sin nx,$$

where

$$\alpha_n = \frac{2}{\pi}\int_0^{\pi} 2\sin nx\, dx = \frac{4}{\pi n}(-\cos \pi n + \cos 0) =$$

$$= \frac{4}{\pi n}[1 - (-1)^n] = \begin{cases} 0, & n = 2k; \\ \dfrac{8}{\pi(2k+1)}, & n = 2k+1, \end{cases}$$

with $k=\overline{0,\infty}$. For the functions $\Theta_n(t)$, we have the following Cauchy problem:

$$\begin{gathered}\Theta_n''(t)+n^2\Theta_n(t)=\alpha_n;\\ \Theta_n(0)=0;\qquad \Theta_n'(0)=0.\end{gathered}\tag{30.47}$$

Seek a particular solution of Eq. (30.47) by the method of indefinite coefficients:

$$\widetilde{\Theta}_n(t)=\beta_n t^0=\beta_n.$$

Then

$$n^2\beta_n=\alpha_n,$$

i.e.,

$$\beta_n=\frac{\alpha_n}{n^2}.$$

Hence,

$$\widetilde{\Theta}_n(t)=\frac{\alpha_n}{n^2},\qquad n=\overline{0,\infty}$$

or

$$\Theta_n(t)=a_n\cos nt+b_n\sin nt+\frac{\alpha_n}{n^2}.$$

From the initial condition for the functions $\Theta_n(t)$ (30.47)

$$\Theta_n(0)=a_n+\frac{\alpha_n}{n^2},$$

we obtain

$$a_n=-\frac{\alpha_n}{n^2},\qquad n=\overline{1,\infty}.$$

From the initial condition for the derivative Θ_n', we obtain $b_n=0$, $n=\overline{1,\infty}$. Then

$$\Theta_n(t)=\frac{\alpha_n}{n^2}(1-\cos nt).$$

As a result, we can write

$$M(x,t)=\sum_{n=1}^{\infty}\Big\{\frac{\alpha_n}{n^2}(1-\cos nt)\Big\}\sin nx.\tag{30.48}$$

Finally,

$$\begin{aligned}u(x,t)&=t^2+\cos t\sin x+\sum_{n=1}^{\infty}\Big\{\frac{4[1-(-1)^n]}{\pi n^3}(1-\cos nt)\Big\}\sin nx=\\ &=t^2+\cos t\sin x+\sum_{k=0}^{\infty}\Big\{\frac{8}{\pi(2k+1)^3}[1-\cos(2k+1)t]\Big\}\sin(2k+1)x,\end{aligned}$$

where the functions $L(x,t)$ and M are defined by relations (30.45) and (30.48), respectively.

31 Spherical waves

Some multidimensional problems can be reduced to a one-dimensional equation.

Earlier we have considered the process of propagation of waves along a half-line. Let us show that the propagation of waves generated by a point source in space admits a similar description.

If for the equation

$$u_{tt} - a^2 \Delta u = 0, \tag{31.1}$$

we specify a boundary condition at a point

$$\text{(a)} \ \ u|_{\vec{r}=0} = \varphi(t) \quad \text{or} \quad \text{(b)} \ \ \frac{\partial u}{\partial r}\Big|_{\vec{r}=0} = \psi(t),$$

a solution of Eq. (31.1) can be sought in the form

$$u(\vec{r}, t) = u(r, t), \quad \text{where} \quad r = |\vec{r}|. \tag{31.2}$$

Write the Laplace operator in spherical coordinates:

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial u}{\partial\theta}\Big). \tag{31.3}$$

Since the function $u(\vec{r}, t)$ (31.2) is independent of θ and φ, Eq. (31.1) takes the form

$$u_{tt} - a^2\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial u}{\partial r}\Big) = 0. \tag{31.4}$$

Note that

$$\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial u}{\partial r}\Big) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru). \tag{31.5}$$

Indeed,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial u}{\partial r}\Big) = \frac{1}{r^2}\Big[2r\frac{\partial u}{\partial r} + r^2\frac{\partial^2 u}{\partial r^2}\Big] = \frac{2}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2};$$
$$\frac{1}{r}\frac{\partial}{\partial r}\Big(\frac{\partial(ru)}{\partial r}\Big) = \frac{1}{r}\frac{\partial}{\partial r}\Big[u + r\frac{\partial u}{\partial r}\Big] = \frac{1}{r}\Big[2\frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}\Big],$$

and so relation (31.5) holds. Then Eq. (31.4) can be written as

$$u_{tt} - \frac{a^2}{r}\frac{\partial^2}{\partial r^2}(ru) = 0. \tag{31.6}$$

Denoting $ru(r, t) = v(r, t)$, we obtain for the function $v(r, t)$ the equation

$$v_{tt} - a^2 v_{rr} = 0, \qquad r \geqslant 0, \tag{31.7}$$

which is identical to that of a vibrating string. The solution of Eq. (31.7) is known:

$$v(r, t) = f_1(t + r/a) + f_2(t - r/a). \tag{31.8}$$

Here f_1 and f_2 are arbitrary smooth functions. Then for $u(r, t)$ we obtain

$$u(r, t) = \frac{f_1(t + r/a) + f_2(t - r/a)}{r}. \tag{31.9}$$

This is the general solution of Eq. (31.1) that describes the propagation of a perturbation produced by a spherically symmetrical (not necessarily point) source located at the origin ($\vec{r} = 0$).

♦ Functions $u(\vec{r}, t)$ of the form (31.9) describe *spherical waves.*

Consider boundary conditions of kind (a) at the point $r = 0$. Let $u(0, t) = \varphi(t)$, where the function $\varphi(t)$ is bounded. Then

$$v(0, t) = ru(r, t)\big|_{r=0} = 0, \qquad f_1(t) = -f_2(t) = f(t),$$

and, hence,

$$u(r, t) = \frac{f(t + r/a) - f(t - r/a)}{r}. \tag{31.10}$$

Pass in (31.10) to the limit as $r \to 0$:

$$\lim_{r\to 0} u(r, t) = \frac{2}{a} f'(t) = u(0, t) = \varphi(t),$$

i.e.,

$$f(t) = \frac{2}{a} \int\limits_0^t \varphi(\tau) d\tau.$$

Hence,

$$u(r, t) = \frac{2}{ar} \int\limits_{t-r/a}^{t+r/a} \varphi(\tau) d\tau. \tag{31.11}$$

A boundary value problem of kind (b) can be solved in a similar way.

♢ The amplitude of a spherical wave (if $|f(x)| \leqslant a$) decreases not slower than inversely proportional to the distance from the center.

32 The Cauchy problem for d'Alembert's equation in space

Let us consider the problem of vibrations propagating in an infinite volume, i.e., the Cauchy problem for the wave equation in the whole space ($\vec{x} \in \mathbb{R}^n$):

$$u_{tt} = a^2 \Delta u + F(\vec{x}, t), \qquad u = u(\vec{x}, t), \quad t > 0, \tag{32.1}$$

$$u(\vec{x}, 0) = \varphi(\vec{x}), \qquad u_t(\vec{x}, 0) = \psi(\vec{x}). \tag{32.2}$$

32.1 The averaging method and Kirchhoff's formula

Theorem 32.1. *Let functions $\varphi(\vec{x})$ and $\psi(\vec{x})$ be, respectively, thrice and twice differentiable in $\mathbb{R}^3$ and a function $f(\vec{x}, t)$ be continuously differentiable with respect to its all variables. Then the classical solution of the Cauchy problem* (32.1), (32.2) ($\vec{x} \in \mathbb{R}^3$) *has the form*

$$u(\vec{x}, t) = \frac{1}{4\pi a^2 t} \int\limits_{|\vec{x}-\vec{y}|=at} \psi(\vec{y}) dS_y + \frac{\partial}{\partial t}\left(\frac{1}{4\pi a^2 t} \int\limits_{|\vec{x}-\vec{y}|=at} \varphi(\vec{y}) dS_y \right) +$$

$$+\frac{1}{4\pi a^2}\int\limits_{|\vec{y}-\vec{x}|\leqslant at}\frac{1}{|\vec{y}-\vec{x}|}f\left(\vec{y},t-\frac{|\vec{y}-\vec{x}|}{a}\right)d\vec{y}. \tag{32.3}$$

Relation (32.3) *is called Kirchhoff's formula.*

Proof. Represent the solution of the problem (32.1), (32.2) in the form

$$u(\vec{x},t)=\tilde{u}(\vec{x},t)+v(\vec{x},t), \tag{32.4}$$

where the functions $\tilde{u}(\vec{x},t)$ and $v(\vec{x},t)$ are solutions of the following problems:

$$\tilde{u}_{tt}=a^2\Delta\tilde{u},\quad t>0; \tag{32.5}$$

$$\tilde{u}(\vec{x},0)=\varphi(\vec{x}),\quad \tilde{u}_t(\vec{x},0)=\psi(\vec{x}) \tag{32.6}$$

and

$$v_{tt}=a^2\Delta v+F(\vec{x},t),\quad t>0; \tag{32.7}$$

$$v(\vec{x},0)=v_t(\vec{x},0)=0. \tag{32.8}$$

The solution of the problem (32.5), (32.6) is based on a trick using quantities averaged over some of the variables. Put $\vec{x}=\vec{y}+\vec{R}$, where

$$\vec{R}=R(\cos\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta),$$

and consider the function

$$\bar{u}(\vec{y},R,t)=\frac{1}{4\pi R^2}\oint\limits_{S_R}\tilde{u}(\vec{y}+\vec{R},t)dS, \tag{32.9}$$

where S_R is a sphere of radius R centered at the origin, $dS=R^2d\Omega$, $d\Omega=\sin\theta d\theta d\varphi$. It can be written in the form

$$\bar{u}(\vec{y},R,t)=\frac{1}{4\pi}\oint\limits_{\Omega}\tilde{u}(\vec{y}+\vec{R},t)d\Omega. \tag{32.10}$$

Integration is carried out over the full solid angle Ω. Obviously, $\bar{u}(\vec{y},0,t)=\tilde{u}(\vec{y},t)$ and

$$\Delta_{\vec{R}}\tilde{u}(\vec{y}+\vec{R},t)=\frac{1}{a^2}\tilde{u}_{tt}(\vec{y}+\vec{R},t). \tag{32.11}$$

The Laplace operator $\Delta_{\vec{R}}$ acts on the variable $\vec{R}$. Integrate the left and right sides of relation (32.11) over a sphere V_R of radius R (variable $\vec{R}$). By Ostrogradskii's formula, we have

$$\int\limits_{V_R}(\Delta_{\vec{R}}\tilde{u})d\vec{R}=\oint\limits_{S_R}\frac{\partial\tilde{u}}{\partial n}dS=\oint\limits_{S_R}R^2\frac{\partial\tilde{u}}{\partial n}d\Omega.$$

Since, for a sphere, $\partial\tilde{u}/\partial n=\partial\tilde{u}/\partial R$, then

$$\int\limits_{V_R}(\Delta_{\vec{R}}\tilde{u})d\vec{R}=R^2\frac{\partial}{\partial n}\oint\tilde{u}\,d\Omega.$$

The last integral, according to (32.10), is equal to $4\pi\bar{u}$. Thus,

$$\int\limits_{V_R} (\Delta_{\vec{R}}\tilde{u})d\vec{R} = 4\pi R^2 \frac{\partial\bar{u}(\vec{y},R,t)}{\partial R}. \tag{32.12}$$

Let us now consider the integral on the right of relation (32.11) over the volume V_R:

$$\frac{1}{a^2}\int\limits_{V_R} \tilde{u}_{tt}(\vec{y}+\vec{R},t)d\vec{R} = \frac{1}{a^2}\frac{\partial^2}{\partial t^2}\int\limits_{V_R} u(\vec{y}+\vec{R},t)d\vec{R} =$$

$$= \frac{1}{a^2}\frac{\partial^2}{\partial t^2}\Big\{\int\limits_0^R R^2 dR \oint\limits_{\Omega} \tilde{u}(\vec{y}+\vec{R},t)d\Omega\Big\}.$$

The inner integral is equal to $4\pi\bar{u}$. Hence,

$$\frac{1}{a^2}\int\limits_{V_R} \tilde{u}_{tt}(\vec{y}+\vec{R},t)d\vec{R} = \frac{4\pi}{a^2}\frac{\partial^2}{\partial t^2}\int\limits_0^R R^2\bar{u}dR. \tag{32.13}$$

Upon equating the right sides of relations (32.12) and (32.13), we obtain

$$R^2\frac{\partial\bar{u}}{\partial R} = \frac{4\pi}{a^2}\frac{\partial^2}{\partial t^2}\int\limits_0^R R^2\bar{u}dR. \tag{32.14}$$

Differentiation of the left and right sides of (32.14) with respect to R yields

$$\frac{\partial}{\partial R}\Big(R^2\frac{\partial\bar{u}}{\partial R}\Big) = \frac{R^2}{a^2}\frac{\partial^2\bar{u}}{\partial t^2}. \tag{32.15}$$

Perform in (32.15) the change $\bar{u} = v/R$ (as in Sec. "Spherical waves"):

$$\frac{\partial^2 v}{\partial R^2} = \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2}. \tag{32.16}$$

Thus,

$$v(R,t) = g\Big(t+\frac{R}{a}\Big) - g\Big(t-\frac{R}{a}\Big), \tag{32.17}$$

since $\bar{u}(0,t) = \tilde{u}(\vec{y},t)$ exists, which is possible only if $v(0,t) = 0$. Hence,

$$\bar{u}(R,t) = \frac{1}{R}\Big[g\Big(t+\frac{R}{a}\Big) - g\Big(t-\frac{R}{a}\Big)\Big], \tag{32.18}$$

and

$$\bar{u}(0,t_0) = \frac{2}{a}g'(t_0) = \tilde{u}(\vec{y},t_0). \tag{32.19}$$

Construct a function L such that

$$L = \frac{\partial}{\partial R}R\bar{u}(R,t) + \frac{1}{a}R\frac{\partial\bar{u}(R,t)}{\partial t} = \frac{2}{a}g'\Big(t+\frac{R}{a}\Big). \tag{32.20}$$

It is obvious that for $R = 0$ and $t = t_0$

$$L\Big|_{\substack{R=0\\t=t_0}} = \frac{2}{a}g'(t_0) = \tilde{u}(\vec{y}, t_0). \tag{32.21}$$

Similarly, for $R = at_0$ and $t = 0$

$$L\Big|_{\substack{t=0\\R=at_0}} = \frac{2}{a}g'(t_0) = \tilde{u}(\vec{y}, t_0),$$

but

$$\bar{u}(\vec{y}, R, 0) = \frac{1}{4\pi}\oint \tilde{u}(\vec{y} + \vec{R}, 0)d\Omega = \frac{1}{4\pi}\oint \varphi(\vec{y} + \vec{R})d\Omega \tag{32.22}$$

and

$$\frac{\partial \bar{u}}{\partial t}\Big|_{t=0} = \frac{1}{4\pi}\oint \psi(\vec{y} + \vec{R})d\Omega. \tag{32.23}$$

Substitute (32.23) and (32.22) into (32.20) and use the explicit form of the function L (32.21):

$$\Big[\frac{\partial}{\partial R}R\frac{1}{4\pi}\oint \varphi(\vec{y} + \vec{R})d\Omega + \frac{1}{a}R\oint \frac{\psi(\vec{y} + \vec{R})d\Omega}{4\pi}\Big]\Big|_{\substack{t=0\\R=at_0}} = \tilde{u}(\vec{y}, t_0).$$

Denote $\vec{y} = \vec{x}$, $t_0 = t$ and write

$$\tilde{u}(\vec{x}, t) = \frac{1}{4\pi}\Big[\frac{\partial}{\partial t}t\oint_{\Omega} \varphi(\vec{x} + \vec{R})d\Omega + t\oint_{\Omega} \psi(\vec{x} + \vec{R})d\Omega\Big], \tag{32.24}$$
$$\vec{R} = at(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta).$$

Thus, we have obtained the solution of the homogeneous wave equation with arbitrary initial conditions in the whole space.

Putting $\vec{y} = \vec{x} + \vec{R}$ and going from the double integral to the surface integral over the sphere $|\vec{x} - \vec{y}| = at$, from relation (32.24), in view of $dS_y = (at)^2 d\Omega$, we arrive at the formula

$$\tilde{u}(\vec{x}, t) = \frac{1}{4\pi a^2}\Big\{\frac{\partial}{\partial t}\Big[\frac{1}{t}\int\limits_{|\vec{x}-\vec{y}|=at} \varphi(\vec{y})dS_y\Big] + \frac{1}{t}\int\limits_{|\vec{x}-\vec{y}|=at} \psi(\vec{y})dS_y\Big\}. \tag{32.25}$$

To solve the nonhomogeneous equation (32.7), we use, as in the one-dimensional case, Duhamel's principle that specifies the solution $v(\vec{x}, t)$ in the form

$$v(\vec{x}, t) = \int\limits_0^t U(\vec{x}, t, \tau)d\tau, \tag{32.26}$$

where $U(\vec{x}, t, \tau)$ is a solution of the Cauchy problem

$$U_{tt} = a^2\Delta U, \tag{32.27}$$
$$U\Big|_{t=\tau} = 0, \quad U_t\Big|_{t=\tau} = F(\vec{x}, \tau).$$

Direct checking, as in proving Statement 29.1, makes us sure that (32.26) is valid. Then, using formula (32.25), we write the solution of the problem (32.27)

$$U(\vec{x},t,\tau)=\frac{1}{4\pi a^2(t-\tau)}\int\limits_{|\vec{x}-\vec{y}|=a(t-\tau)}F(\vec{y},\tau)dS_y.$$

Substituting this relation into (32.26) yields

$$v(\vec{x},t)=\frac{1}{4\pi a^2}\int\limits_0^t\frac{d\tau}{t-\tau}\int\limits_{|\vec{x}-\vec{y}|=a(t-\tau)}F(\vec{y},\tau)dS_y,$$

from which, with the change $t-\tau=r/a$, we find

$$v(\vec{x},t)=\frac{1}{4\pi a^2}\int\limits_0^{at}\Big[\int\limits_{|\vec{x}-\vec{y}|=r}\frac{F(\vec{y},t-r/a)}{r}dS_y\Big]dr=$$
$$=\frac{1}{4\pi a^2}\int\limits_{|\vec{x}-\vec{y}|<at}\frac{F(\vec{y},t-|\vec{x}-\vec{y}|/a)}{|\vec{x}-\vec{y}|}d\vec{y}. \qquad (32.28)$$

Substitution of (32.25) and (32.28) into (32.4) yields Kirchhoff's formula (32.3), Q. E. D.

◊ Kirchhoff's formula (32.3) holds true if the space dimension exceeds three, i.e., $\vec{x}\in\mathbb{R}^n$, $n\geqslant 3$.

Example 32.1. Solve the Cauchy problem

$$u_{tt}=a^2\Delta u+f(\vec{x},t),\qquad \vec{x}\in\mathbb{R}^3, \qquad (32.29)$$
$$u\Big|_{t=0}=\varphi(\vec{x}),\qquad u_t\Big|_{t=0}=\psi(\vec{x}), \qquad (32.30)$$

for

$$a^2=8,\quad f(\vec{x},t)=x_1^2t^2,\quad \varphi(\vec{x})=x_2^2,\quad \psi(\vec{x})=x_3^2. \qquad (32.31)$$

Solution. The solution of the problem is given by Kirchhoff's formula

$$u(\vec{x},t)=\frac{1}{4\pi a^2}\int\limits_{|\vec{y}-\vec{x}|\leqslant at}\frac{1}{|\vec{y}-\vec{x}|}f\Big(\vec{y},t-\frac{|\vec{y}-\vec{x}|}{a}\Big)d\vec{y}+$$
$$+\frac{1}{4\pi a^2 t}\int\limits_{|\vec{y}-\vec{x}|=at}\psi(\vec{y})dS'+\frac{1}{4\pi a^2}\frac{\partial}{\partial t}\Big[\frac{1}{t}\int\limits_{|\vec{y}-\vec{x}|=at}\varphi(\vec{y})dS'\Big]. \qquad (32.32)$$

The first integral in (32.32) is a triple integral over a sphere of radius at

$$(\vec{x}-\vec{y})^2\leqslant(at)^2,$$

while the other two are surface integrals over a sphere $(\vec{x}-\vec{y})^2=a^2t^2$. To calculate them, it is convenient to go to a spherical coordinate system with the origin at the point $\vec{x}=(x_1,x_2,x_3)$:

$$\begin{aligned}y_1&=x_1+\rho\sin\theta\cos\varphi,\\ y_2&=x_2+\rho\sin\theta\sin\varphi,\\ y_3&=x_3+\rho\cos\varphi.\end{aligned} \qquad (32.33)$$

Calculate the first integral in (32.32). In view of (32.31), we have

$$I_1 = \frac{1}{4\pi a^2} \int\limits_{|\vec{y}-\vec{x}|\leqslant at} \frac{y_1^2\big[t - \sqrt{(y_1-x_1)^2+(y_2-x_2)^2+(y_3-x_3)^2}/a\big]^2}{\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+(y_3-x_3)^2}} d\vec{y} \tag{32.34}$$

and, going to the coordinates (32.33), we can write

$$I_1 = \frac{1}{4\pi a^2} \int\limits_0^{at} \rho^2 d\rho \int\limits_0^{\pi} \sin\theta d\theta \int\limits_0^{2\pi} d\varphi \frac{(x_1+\rho\sin\theta\cos\varphi)^2(t-\rho/a)^2}{\rho}. \tag{32.35}$$

Successive calculation of the integrals

$$\begin{gathered}
\int\limits_0^{2\pi}(x_1+\rho\sin\theta\cos\varphi)^2 d\varphi = 2\pi x_1^2 + \pi\rho^2\sin^2\theta, \\
\int\limits_0^{\pi}(2\pi x_1^2+\pi\rho^2\sin^2\theta)\sin\theta d\theta = 4\pi\Big(x_1^2+\frac{\rho^2}{3}\Big), \\
4\pi\int\limits_0^{at}\rho\Big(t-\frac{\rho}{a}\Big)^2\Big(x_1^2+\frac{\rho^2}{3}\Big)d\rho = 4\pi a^2\Big[\frac{x_1^2t^4}{12}+\frac{a^2t^6}{180}\Big]
\end{gathered} \tag{32.36}$$

yields

$$\widetilde{I}_1 = \frac{1}{4\pi a^2}4\pi a^2\Big[\frac{x_1^2t^4}{12}+\frac{a^2t^6}{180}\Big] = \frac{x_1^2t^4}{12}+\frac{a^2t^6}{180},$$

and, since $a^2 = 8$,

$$\widetilde{I}_1 = \frac{x_1^2t^4}{12}+\frac{2}{45}t^6. \tag{32.37}$$

Calculate the second integral in (32.32). In view of (32.31), we have

$$\begin{aligned}
I_2 &= \frac{1}{4\pi a^2 t}\int\limits_{|\vec{y}-\vec{x}|=at} y_1^2 dS' = \frac{1}{4\pi a^2 t}\int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi}(x_3+at\cos\theta)^2(at)^2\sin\theta d\theta = \\
&= \frac{t}{2}\int\limits_0^{\pi}(x_3+at\cos\theta)^2\sin\theta d\theta = t\Big(x_1^2+\frac{a^2t^2}{3}\Big),
\end{aligned}$$

and, finally,

$$\widetilde{I}_2 = t\Big(x_3^2+\frac{8t^2}{3}\Big). \tag{32.38}$$

The third integral in (32.32) is calculated in a similar way:

$$I_3 = \int\limits_{|\vec{y}-\vec{x}|=at} y_2^2 dS' = \int\limits_0^{2\pi} d\varphi\int\limits_0^{\pi}(x_2+at\sin\theta\sin\varphi)^2(at)^2\sin\theta d\theta = 4\pi(at)^2\Big(x_2^2+\frac{a^2t^2}{3}\Big),$$

whence

$$\widetilde{I}_3 = \frac{1}{4\pi a^2}\frac{\partial}{\partial t}\Big(\frac{1}{t}I_3\Big) = \frac{\partial}{\partial t}\Big[t\Big(x_2^2 + \frac{a^2t^2}{3}\Big)\Big] = x_2^2 + a^2t^2$$

or, for $a^2 = 8$,

$$\widetilde{I}_3 = x_2^2 + 8t^2. \tag{32.39}$$

Substitution of (32.37), (32.38), and (32.39) into (32.32) yields

$$u(x_1, x_2, x_3, t) = \frac{x_1^2t^4}{12} + \frac{2}{45}t^6 + t\Big(x_3^2 + \frac{8t^2}{3}\Big) + (x_2^2 + 8t^2).$$

Direct checking makes us sure that the resulting relation is correct.

32.2 The descent method and Poisson's formula

Theorem 32.2. *The classical solution of the Cauchy problem* (32.1), (32.2) *for* $\vec{x} \in \mathbb{R}^2$ *has the form*

$$u(\vec{x}, t) = \frac{1}{2\pi a}\int\limits_{|\vec{x}-\vec{y}|<at}\frac{\psi(\vec{y})d\vec{y}}{\sqrt{a^2t^2 - |\vec{x}-\vec{y}|^2}} + \frac{1}{2\pi a}\frac{\partial}{\partial t}\Big(\int\limits_{|\vec{x}-\vec{y}|<at}\frac{\varphi(\vec{y})d\vec{y}}{\sqrt{a^2t^2 - |\vec{x}-\vec{y}|^2}}\Big) +$$
$$+\frac{1}{2\pi a}\int\limits_0^t d\tau\int\limits_{|\vec{y}-\vec{x}|<a(t-\tau)}\frac{F(\vec{y},\tau)d\vec{y}}{\sqrt{a^2(t-\tau)^2 - |\vec{y}-\vec{x}|^2}} + \frac{1}{2\pi a}\int\limits_{|\vec{y}-\vec{x}|<at}\frac{\psi(\vec{y})d\vec{y}}{\sqrt{a^2t^2 - |\vec{x}-\vec{y}|^2}}, \tag{32.40}$$

where $\varphi(\vec{x})$ *and* $\psi(\vec{x})$ *are, respectively, twice and singly differentiable functions in* $\mathbb{R}^2$ *and the function* $f(\vec{x}, t)$ *is continuously differentiable with respect to all its variables.*

Relation (32.40) *is called Poisson's formula of the Cauchy problem for the wave equation in* $\mathbb{R}^2$.

Proof. As in the proof of Theorem 32.1, use the reduction of the problem (32.1), (32.2) in the form (32.4)–(32.8), putting here $\vec{x} \in \mathbb{R}^2$. To find the function $\tilde{u}(\vec{x}, t)$, we apply the descent method, proposed by Hadamard, which allows us, using a known solution in $\mathbb{R}^{n+1}$, to construct a solution of a problem of lower dimension ($\mathbb{R}^{k+1}$ with $k < n$). The essence of the method is that the solution (32.25) can be considered as a solution in $\mathbb{R}^2$, provided that the initial conditions depend only on x_1 and x_2, i.e., $\varphi(\vec{x}) = \varphi(x_1, x_2)$, and $\psi(\vec{x}) = \psi(x_1, x_2)$, and $\partial\tilde{u}/\partial x_3 = 0$.

By the formula of reducing surface integrals to double ones, known from calculus, we have

$$\int\limits_S f(y_1, y_2, y_3)dS = \int\limits_E f(y_1, y_2, y_3(y_1, y_2))\sqrt{1 + \Big(\frac{\partial y_3}{\partial y_1}\Big)^2 + \Big(\frac{\partial y_3}{\partial y_2}\Big)^2}dy_1dy_2,$$

where E is the projection of the surface S on the plane $y_3 = 0$. Apply this formula to (32.25) and take into account that the surface S, given by $|\vec{x} - \vec{y}| = at$ or $x_3 - y_3 = \pm\sqrt{a^2t^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2}$, is projected on the plane $y_3 = x_3$ two times (the top and bottom hemisphere). Also take into account that

$$\frac{\partial y_3}{\partial y_1} = \pm\frac{x_1 - y_1}{\sqrt{a^2t^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2}},$$
$$\frac{\partial y_3}{\partial y_2} = \pm\frac{x_2 - y_2}{\sqrt{a^2t^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2}},$$
$$\sqrt{1 + \Big(\frac{\partial y_3}{\partial y_1}\Big)^2 + \Big(\frac{\partial y_3}{\partial y_2}\Big)^2} = \frac{t}{\sqrt{a^2t^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2}}.$$

As a result, for $\vec{x}, \vec{y} \in \mathbb{R}^2$, we obtain

$$\tilde{u}(\vec{x}, t) = \frac{1}{2\pi a}\int\limits_{|\vec{x}-\vec{y}|<at} \frac{\psi(\vec{y})d\vec{y}}{\sqrt{a^2t^2 - |\vec{x} - \vec{y}|^2}} + \frac{1}{2\pi a}\frac{\partial}{\partial t}\int\limits_{|\vec{x}-\vec{y}|<at} \frac{\varphi(\vec{y})d\vec{y}}{\sqrt{a^2t^2 - |\vec{x} - \vec{y}|^2}}. \qquad (32.41)$$

Now, applying Duhamel's principle, for finding the function $v(\vec{x}, t)$ in $\mathbb{R}^2$ (see Theorem 32.1), in view of (32.41), we obtain

$$v(\vec{x}, t) = \frac{1}{2\pi a}\int\limits_0^t d\tau \int\limits_{|\vec{x}-\vec{y}|<a(t-\tau)} \frac{F(\vec{y}, \tau)d\vec{y}}{\sqrt{a^2(t-\tau)^2 - |\vec{x} - \vec{y}|^2}}. \qquad (32.42)$$

Substituting (32.41) and (32.42) into (32.4), we arrive at Poisson's formula (32.40), Q. E. D.

♢ Applying the descent method to Poisson's formula, we obtain d'Alembert's formula (28.2).

♢ The stability of the solution of the Cauchy problem (32.1), (32.2) in $\mathbb{R}^2$ and $\mathbb{R}^3$ (Kirchhoff's and Poisson's formulas, respectively) is proved in the same manner as the stability of d'Alembert's formula.

♢ Kirchhoff's, Poisson's, and d'Alembert's formulas describe spherical, cylindrical, and plane waves, respectively.

Example 32.2. Solve the Cauchy problem

$$u_{tt} = a^2\Delta u + f(\vec{x}, t), \qquad \vec{x} \in \mathbb{R}^2, \qquad (32.43)$$
$$u\Big|_{t=0} = \varphi(\vec{x}), \qquad u_t\Big|_{t=0} = \psi(\vec{x}), \qquad (32.44)$$

for

$$a = 1,\; f(\vec{x}, t) = 6x_1x_2t,\; \varphi(\vec{x}) = x_1^2 - x_2^2,\; \psi(\vec{x}) = x_1x_2. \qquad (32.45)$$

Solution. The solution of the problem is given by Poisson's formula (32.40)

$$u(\vec{x}, t) = \frac{1}{2\pi a}\int\limits_0^t d\tau \int\limits_{|\vec{y}-\vec{x}|<a(t-\tau)} \frac{f(\vec{y}, \tau)d\vec{y}}{\sqrt{a^2(t-\tau)^2 - |\vec{y} - \vec{x}|^2}} +$$
$$+\frac{1}{2\pi a}\int\limits_{|\vec{y}-\vec{x}|<at} \frac{\psi(\vec{y})d\vec{y}}{\sqrt{a^2t^2 - |\vec{x} - \vec{y}|^2}} + \frac{1}{2\pi a}\frac{\partial}{\partial t}\int\limits_{|\vec{y}-\vec{x}|<at} \frac{\varphi(\vec{y})d\vec{y}}{\sqrt{a^2t^2 - |\vec{x} - \vec{y}|^2}}. \qquad (32.46)$$

All integrals in (32.46) are double integrals over a circle of radius at

$$(y_1 - x_1)^2 + (y_2 - x_2)^2 \leqslant (at)^2.$$

To calculate these integrals, it is convenient to go to a polar coordinate system with the origin at the point $\vec{x} = (x_1, x_2)$

$$y_1 = x_1 + \rho\cos\varphi, \qquad y_2 = x_2 + \rho\sin\varphi. \tag{32.47}$$

Then for the first integral of (32.46), in view of (32.45) and (32.47), we have

$$I_1 = \int\limits_{|\vec{y}-\vec{x}|<a(t-\tau)} \frac{6y_1y_2\tau\, dy_1dy_2}{\sqrt{a^2(t-\tau)^2-(y_1-x_1)^2-(y_2-x_2)^2}} =$$
$$= 6\tau\int\limits_0^{a(t-\tau)}\rho d\rho\int\limits_0^{2\pi}d\varphi\frac{(x_1+\rho\cos\varphi)(x_2+\rho\sin\varphi)}{\sqrt{a^2(t-\tau)^2-\rho^2}}. \tag{32.48}$$

Successive calculation of the integrals

$$\int\limits_0^{2\pi}(x_1+\rho\cos\varphi)(x_2+\rho\sin\varphi)d\varphi = 2\pi x_1x_2,$$
$$\int\limits_0^{a(t-\tau)}\frac{\rho d\rho}{\sqrt{a^2(t-\tau)^2-\rho^2}} = -\sqrt{a^2(t-\tau)^2-\rho^2}\Big|_0^{a(t-\tau)} = a(t-\tau)$$

yields

$$I_1 = 6\tau 2\pi x_1x_2a(t-\tau). \tag{32.49}$$

The quantity I_1, according to Poisson's formula (32.46), should also be integrated with respect to τ. Then

$$\widetilde{I}_1 = \frac{1}{2\pi a}\int\limits_0^t I_1d\tau = 6x_1x_2\int\limits_0^t\tau(t-\tau)d\tau = x_1x_2t^3. \tag{32.50}$$

The rest two integrals are calculated in a similar manner. Indeed,

$$I_2 = \frac{1}{2\pi a}\int\limits_{|\vec{y}-\vec{x}|\leqslant at}\frac{y_1y_2dy_1dy_2}{\sqrt{a^2t^2-(y_1-x_1)^2-(y_2-x_2)^2}} =$$
$$= \frac{1}{2\pi a}\int\limits_0^{at}\frac{\rho d\rho}{\sqrt{a^2t^2-\rho^2}}\int\limits_0^{2\pi}(x_1+\rho\cos\varphi)(x_2+\rho\sin\varphi)d\varphi =$$
$$= \frac{x_1x_2}{a}\int\limits_0^{at}\frac{\rho d\rho}{\sqrt{a^2t^2-\rho^2}} = x_1x_2t \tag{32.51}$$

and

$$I_3 = \int\limits_{|\vec{y}-\vec{x}|<at} \frac{(y_1^2 - y_2^2)dy_1dy_2}{\sqrt{a^2t^2 - (y_1-x_1)^2 - (y_2-x_2)^2}} =$$

$$= \int\limits_0^{at} \frac{\rho d\rho}{\sqrt{a^2t^2-\rho^2}} \int\limits_0^{2\pi} [(x_1+\rho\cos\varphi)^2 - (x_2+\rho\sin\varphi)^2]d\varphi =$$

$$= 2\pi(x_1^2 - x_2^2)\int\limits_0^{at} \frac{\rho d\rho}{\sqrt{a^2t^2-\rho^2}} = 2\pi(x_1^2-x_2^2)at,$$

whence

$$\widetilde{I}_3 = \frac{1}{2\pi a}\frac{\partial}{\partial t}[I_3] = \frac{\partial}{\partial t}\big[t(x_1^2-x_2^2)\big] = x_1^2 - x_2^2. \tag{32.52}$$

Substituting (32.50), (32.51), and (32.52) into (32.46), we get

$$u(\vec{x},t) = x_1x_2t^3 + x_1x_2t + x_1^2 - x_2^2.$$

Direct checking makes us sure that the resulting relation is correct.

CHAPTER 7
Parabolic equations

33 The Cauchy problem for the one-dimensional heat equation. Green's function of the Cauchy problem

33.1 The Cauchy problem. The method of separation of variables

Let us consider an infinitely thin, long homogeneous heat conducting rod whose lateral surface is thermally insulated. The heat equation for this case has the form (see Sec. "The heat equation for a rod")

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad t \geqslant 0, \quad -\infty < x < \infty. \tag{33.1}$$

$\Diamond$ For a long rod, the heat exchange processes occurring in its middle part are affected only by the initial temperature distribution and the boundary conditions, while the heat exchange at the rod ends can be neglected. Assume that the initial temperature distribution is given by

$$u(x, 0) = u|_{t=0} = \varphi(x). \tag{33.2}$$

$\blacklozenge$ The problem of finding a solution of Eq. (33.1) with condition (33.2) is called the Cauchy problem.

Seek a particular solution of Eq. (33.1) in the form

$$u(x, t) = T(t)X(x). \tag{33.3}$$

Substitution of this expression into (33.1) yields

$$\frac{1}{a^2}\frac{T'}{T} = \frac{X''}{X} = \lambda, \qquad T' - a^2\lambda T = 0, \qquad X'' - \lambda X = 0.$$

The general solution of the equation for $T(t)$ has the form

$$T(t) = Ce^{\lambda a^2 t}. \tag{33.4}$$

Since there is no heat source in the rod, there is no x at which the rod temperature would be infinite, i.e., $|u(x, t)| < \infty$. Hence, $|T(t)| < \infty$ and $|X(x)| < \infty$.

Thus, λ can be only negative. Put $\lambda = -\omega^2$. Then

$$T(t) = Ce^{-\omega^2 a^2 t}, \tag{33.5}$$

and for the function $X(x)$ we have the equation

$$X'' + \omega^2 X = 0$$

whose general solution has the form

$$X(x) = A_\omega \cos \omega x + B_\omega \sin \omega x.$$

Introducing a complex coefficient C_ω, we obtain

$$X(x) = C_\omega e^{i\omega x} + C_\omega^* e^{-i\omega x},$$

and, correspondingly,

$$u_\omega(x,t) = e^{-\omega^2 a^2 t}(A_\omega \cos\omega x + B_\omega \sin\omega x) = \\ = e^{-\omega^2 a^2 t}(C_\omega e^{i\omega x} + C_\omega^* e^{-i\omega x}).$$

Integration of this equation with respect to ω from zero to infinity yields

$$u(x,t) = \int\limits_0^\infty d\omega\, e^{-\omega^2 a^2 t}(A_\omega \cos\omega x + B_\omega \sin\omega x)$$

or

$$u(x,t) = \int\limits_{-\infty}^{\infty} d\omega\, e^{-\omega^2 a^2 t} C_\omega e^{i\omega x}. \tag{33.6}$$

For $t = 0$, condition (33.2) must be fulfilled:

$$u(x,0) = \int\limits_{-\infty}^{\infty} d\omega\, C_\omega e^{i\omega x} = \varphi(x) = \int\limits_{-\infty}^{\infty} d\omega\, \alpha_\omega e^{i\omega x},$$

where

$$\alpha_\omega = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \varphi(y) e^{-i\omega y} dy$$

are Fourier coefficients of the function $\varphi(x)$. From the equality of two Fourier integrals it follows that $C_\omega = \alpha_\omega$, i.e.,

$$u(x,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} d\omega \int\limits_{-\infty}^{\infty} dy\, \varphi(y) e^{-\omega^2 a^2 t} e^{i\omega(x-y)}. \tag{33.7}$$

33.2 The Green's function of the Cauchy problem

♦ A generalized function $G(x,y,t)$ which satisfies the conditions

$$\frac{\partial G(x,y,t)}{\partial t} = a^2 \frac{\partial^2 G(x,y,t)}{\partial x^2}, \tag{33.8}$$

$$G(x,y,0) = \delta(x-y). \tag{33.9}$$

is called the Green's function of the Cauchy problem for the heat equation (33.1).

Statement 33.1. *For the Green's function, the representation*

$$G(x,y,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-\omega^2 a^2 t} e^{i\omega(x-y)} d\omega \tag{33.10}$$

is valid.

In view of (33.10), relation (33.7) can be written as

$$u(x,t) = \int\limits_{-\infty}^{\infty} G(x,y,t)\varphi(y)dy. \tag{33.11}$$

Since the function $u(x,t)$ satisfies Eq. (33.1), from (33.11) it follows that the function $G(x,y,t)$ satisfies Eq. (33.8).

For $t = 0$ we get

$$G(x,y,0) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{i\omega(x-y)}d\omega = \delta(x-y).$$

Here we have used the representation of the δ-function by a Fourier integral (see Sec. "Fourier transforms of slowly rising generalized functions" of Part II). The validity of Eq. (33.8) can be verified by direct substitution of (33.10) into (33.8), which proves the statement.

From definition (33.10), we can easily derive the following property of the function $G(x,y,t)$:

Property 1. The relation

$$\int\limits_{-\infty}^{\infty} G(x,y,t)dx = 1 \tag{33.12}$$

is valid.

Proof. Indeed, we have

$$\int\limits_{-\infty}^{\infty} G(x,y,t)dx = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} dx \int\limits_{-\infty}^{\infty} e^{-\omega^2a^2t}e^{i\omega(x-y)}d\omega =$$

$$= \int\limits_{-\infty}^{\infty} d\omega\, e^{-\omega^2a^2t}e^{i\omega y}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{i\omega x}dx =$$

$$= \int\limits_{-\infty}^{\infty} d\omega\, e^{-\omega^2a^2t}e^{i\omega y}\delta(\omega) = e^{-\omega^2a^2t}e^{i\omega y}\Big|_{\omega=0} = 1,$$

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Property 2. The relation

$$\int\limits_{-\infty}^{\infty} G(x,y,t)dy = 1$$

is valid.

Proof is similar to that of the preceding property.

Property 3. The relation

$$G(x,y,t) = \frac{1}{2\sqrt{\pi t a^2}}e^{-[(x-y)/(2a\sqrt{t})]^2} \tag{33.13}$$

is valid.

Proof. Calculate the integral with respect to ω in (33.10). For doing this, represent the function $G(x, y, t)$ in the form

$$G(x,y,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-a^2\omega^2 t}[\cos\omega(x-y) + i\sin\omega(x-y)]d\omega.$$

The integral of the second term vanishes because of the oddness of the sine function. Thus,

$$G(x,y,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-a^2\omega^2 t}\cos\omega(x-y)d\omega.$$

Perform in this integral the change of variables

$$\omega = \frac{z}{a\sqrt{t}}, \qquad d\omega = \frac{dz}{a\sqrt{t}}.$$

Then

$$G(x,y,t) = \frac{1}{\pi}\int\limits_{0}^{\infty} e^{-z^2}\cos\varkappa z\frac{dz}{a\sqrt{t}}, \qquad \varkappa = \frac{x-y}{a\sqrt{t}}. \tag{33.14}$$

Denoting

$$J(\varkappa) = \int\limits_{0}^{\infty} e^{-z^2}\cos\varkappa z dz,$$

we find

$$\frac{dJ(\varkappa)}{d\varkappa} = \int\limits_{0}^{\infty} e^{-z^2}(-z)\sin\varkappa z dz.$$

Calculate the second integral by parts, putting $U = \sin\varkappa z$ and $dV = -ze^{-z^2}dz$. Then $dU = \varkappa\cos\varkappa z dz$ and $V = e^{-z^2}/2$, and

$$\frac{dJ(\varkappa)}{d\varkappa} = \frac{1}{2}e^{-z^2}\sin\varkappa z\Big|_0^\infty - \frac{\varkappa}{2}\int\limits_{0}^{\infty} e^{-z^2}\cos\varkappa z dz.$$

The term outside the integral vanishes and the integral is equal to $J(\varkappa)$. Thus, for $J(\varkappa)$ we obtain the differential equation

$$\frac{dJ(\varkappa)}{d\varkappa} = -\frac{\varkappa}{2}J(\varkappa),$$

whose solution is the function

$$J(\varkappa) = Ce^{-\varkappa^2/4}.$$

The constant multiplier C is determined from the condition

$$J(0) = \int\limits_{0}^{\infty} e^{-z^2}dz.$$

Perform the change of variables $z^2 = t$, $z = \sqrt{t}$, and $dz = dt/(2\sqrt{t})$. Then

$$J(0) = \frac{1}{2}\int\limits_0^\infty e^{-t}t^{-1/2}dt = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2},$$

i.e.,

$$C = \frac{\sqrt{\pi}}{2}, \qquad J(\varkappa) = \frac{\sqrt{\pi}}{2}e^{-\varkappa^2/4}.$$

Since,

$$G(x, y, t) = \frac{1}{\pi a\sqrt{t}}J(\varkappa),$$

we come to (33.13). Thus, Property 5 is proved.

Corollary. For processes described by the heat equation, the rate of heat transfer is equal to infinity.

Proof. Assume that the initial temperature distribution in a rod has the form

$$\varphi(x) = \begin{cases} f(x), & x \in [a, b] \\ 0, & x \notin [a, b], \end{cases} \tag{33.15}$$

where $f(x)$ is a continuous function. Then, according to (33.10),

$$u(x, t) = \int\limits_{-\infty}^{\infty} G(x, y, t)\varphi(y)dy = \int\limits_a^b G(x, y, t)f(y)dy. \tag{33.16}$$

From the explicit form of Green's function (33.13) it follows that the function (33.16) is infinitely differentiable with respect to x and t at $t \neq 0$. Moreover, $u(x, t) \neq 0$ at $t \neq 0$ (no matter how small) for any x (no matter how great), while the initial temperature $u(x, 0) = \varphi(x)$ (33.15) is equal to zero for $x \notin [a, b]$. Hence, the rate of heat transfer is infinite.

◊ For other parabolic equations, the conclusion that the rate of transfer of an "interaction" is infinite holds true.

Example 33.1. Find the form of the Klein–Gordon equation (14.7) in the nonrelativistic limit as $c \to \infty$.

Solution. In the Klein–Gordon equation

$$\frac{\hbar^2}{c^2}\Psi_{tt} - \hbar^2\Delta\Psi + m_0^2c^2\Psi = 0, \tag{33.17}$$

perform the change

$$\Psi(\vec{x}, t) = \varphi(\vec{x}, t)\exp\Big(-i\frac{m_0c^2}{\hbar}t\Big). \tag{33.18}$$

In view of the relations

$$\Psi_t = \Big(\varphi_t - i\frac{m_0c^2}{\hbar}\Big)e^{-im_0c^2t/\hbar},$$

$$\Psi_{tt} = \Big(\varphi_{tt} - 2i\frac{m_0c^2}{\hbar}\varphi_t - \frac{m_0^2c^4}{\hbar^2}\Big)e^{-im_0c^2t/\hbar},$$

we obtain

$$\frac{\hbar^2}{c^2}\varphi_{tt} + 2m_0(-i\hbar\varphi_t) + (-i\hbar)^2\Delta\varphi = 0.$$

In the limit, as $c \to \infty$, we have

$$(-i\hbar)\varphi_t - \frac{\hbar^2\Delta\varphi}{2m_0} = O\Big(\frac{1}{c^2}\Big). \qquad (33.19)$$

Thus, as $c \to \infty$, the Klein–Gordon equation goes into a Schrödinger equation.

Green's function of the Cauchy problem, $G(x, y, t)$, allows us to find a particular solution of the nonhomogeneous heat equation.

Property 4 (Duhamel's principle). The solution of the Cauchy problem for the nonhomogeneous heat equation with a homogeneous initial condition

$$\frac{\partial\bar{u}}{\partial t} = a^2\frac{\partial^2\bar{u}}{\partial x^2} + f(x, t), \qquad \bar{u}|_{t=0} = 0$$

has the form

$$\bar{u}(x, t) = \int\limits_0^t d\tau \int\limits_{-\infty}^{\infty} dy\, G(x, y, t-\tau)f(y, \tau), \qquad (33.20)$$

where $G(x, y, t)$ is Green's function (33.13) of the heat equation.

♦ Relation (33.20) is called Duhamel's formula for the one-dimensional heat equation.

Proof. Indeed, from (33.20) we have

$$\frac{\partial\bar{u}}{\partial t} = \int\limits_{-\infty}^{\infty} G(x, y, 0)f(y, t)dy + \int\limits_0^t d\tau \int\limits_{-\infty}^{\infty} dy\frac{\partial G(x, y, t-\tau)}{\partial(t-\tau)}f(y, \tau).$$

Taking into account that $G(x, y, 0) = \delta(x - y)$ and denoting $T = t - \tau$, we get

$$\frac{\partial\bar{u}}{\partial t} = f(x, t) + \int\limits_0^t d\tau \int\limits_{-\infty}^{\infty} dy\frac{\partial G(x, y, T)}{\partial T}f(y, \tau).$$

Similarly,

$$\frac{\partial^2\bar{u}}{\partial x^2} = \int\limits_0^t d\tau \int\limits_{-\infty}^{\infty} dy\frac{\partial^2 G(x, y, T)}{\partial x^2}f(y, \tau).$$

Substituting this into Eq. (33.1), we find

$$\frac{\partial\bar{u}}{\partial t} - \frac{\partial^2\bar{u}}{\partial x^2} = f(x, t) + \int\limits_0^t d\tau \int\limits_{-\infty}^{\infty} dy\Big(\frac{\partial G}{\partial T} - a^2\frac{\partial^2 G(x, y, T)}{\partial x^2}\Big)f(y, t).$$

Since the bracketed expression is equal to zero, we have

$$\frac{\partial \bar{u}}{\partial t} - \frac{\partial^2 \bar{u}}{\partial x^2} = f(x,t),$$

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Property 5 (preservation of evenness). The evenness of an initial state is preserved; that is, if a function $u(x,0) = \varphi(x)$ is even (odd):

$$\varphi(-x) = \pm\varphi(x), \tag{33.21}$$

the function $u(x,t)$ will be even (odd) for all $t \geqslant 0$:

$$u(-x,t) = \pm u(x,t). \tag{33.22}$$

Proof. From the explicit form of the function $G(x,y,t)$ (33.13) it follows that

$$G(-x,-y,t) = G(x,y,t). \tag{33.23}$$

For the solution of the Cauchy problem (33.1), (33.2), according to (33.11), we have

$$u(-x,t) = \int\limits_{-\infty}^{\infty} G(-x,y,t)\varphi(y)dy.$$

Making in this integral the change of variables $y \to -y$ and using relations (33.23) and (33.21), we get

$$u(-x,t) = \int\limits_{-\infty}^{\infty} G(-x,-y,t)\varphi(-y)dy =$$

$$= \int\limits_{-\infty}^{\infty} G(x,y,t)\varphi(-y)dy = \pm u(x,t).$$

This proves the statement.

Corollary. If

$$\varphi(-x) = -\varphi(x), \tag{33.24}$$

then

$$u(0,t) = 0. \tag{33.25}$$

Proof immediately follows from (33.22).

34 Mixed problems for the one-dimensional homogeneous heat equation on an interval

Let us consider the mixed problem for the nonhomogeneous heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad t > 0, \quad 0 < x < l, \tag{34.1}$$

$$\begin{gathered} u|_{t=0} = \varphi(x), \qquad \frac{\partial u}{\partial x}\Big|_{x=0} = k_1\{u|_{x=0} - \mu_1(t)\}, \\ \frac{\partial u}{\partial x}\Big|_{x=l} = k_2\{u|_{x=l} - \mu_2(t)\} \end{gathered} \tag{34.2}$$

Assume that the compatibility conditions

$$\frac{\partial u}{\partial x}\Big|_{\substack{x=0\\t=0}} = k_1\big[\varphi(x) - \mu_1(t)\big]\Big|_{\substack{x=0,\\t=0}} \quad \frac{\partial u}{\partial x}\Big|_{\substack{x=l\\t=0}} = k_2\big[\varphi(x) - \mu_2(t)\big]\Big|_{\substack{x=l\\t=0}}$$

are fulfilled. Below we shall restrict ourselves to a detailed solution of this problem with boundary conditions of the first kind only. The procedure of solution of the problem (34.1), (34.2) for a more general situation was presented in the section devoted to the Sturm–Liouville problem.

34.1 The homogeneous mixed problem

Let us consider the mixed problem for the homogeneous heat equation with homogeneous boundary conditions

$$\begin{gathered} u_t = a^2 u_{xx}, \quad t \geqslant 0, \quad 0 \leqslant x \leqslant l, \\ u|_{t=0} = \varphi(x), \qquad u|_{x=0} = 0, \qquad u|_{x=l} = 0. \end{gathered} \tag{34.3}$$

Seek a particular solution of Eq. (34.3) in the form

$$u(x,t) = X(x)T(t). \tag{34.4}$$

Substitution of (34.4) into (34.3) gives us

$$\frac{1}{a^2}\frac{\dot T}{T} = \frac{X''}{X} = \lambda$$

or

$$\dot T - \lambda a^2 T = 0, \quad X'' - \lambda X = 0, \quad X(0) = X(l) = 0.$$

The solution of the Sturm–Liouville problem for the function $X(x)$ has been obtained in Example III.2.2:

$$X_n(x) = A_n \sin\frac{\pi n x}{l}, \qquad \lambda_n = -\Big(\frac{\pi n}{l}\Big)^2. \tag{34.5}$$

Substituting λ_n into the equation for T, we find

$$T_n(t) = B_n \exp\Big\{-\Big(\frac{\pi n a}{l}\Big)^2 t\Big\}.$$

Thus, the solution of Eq. (34.3) has the form

$$u(x,t)=\sum_{n=1}^{\infty}\bar{A}_n\exp\Big\{-\Big(\frac{\pi na}{l}\Big)^2 t\Big\}\sin\frac{\pi nx}{l}, \tag{34.6}$$

where $\bar{A}_n=A_nB_n$. From the initial conditions we find

$$\bar{A}_n=\alpha_n, \tag{34.7}$$

where α_n are Fourier coefficients of the function $\varphi(x)$:

$$\varphi(x)=\sum_{n=1}^{\infty}\alpha_n\sin\frac{\pi nx}{l},\qquad \alpha_n=\frac{2}{l}\int\limits_0^l\varphi(y)\sin\frac{\pi ny}{l}dy. \tag{34.8}$$

♦ A generalized function $g(x,y,t)$ is called the Green's function (fundamental solution) of the mixed problem (34.1), (34.2) for the heat equation if this function satisfies the equation

$$g_t(x,y,t)=a^2g_{xx}(x,y,t), \tag{34.9}$$

the initial condition

$$g(x,y,0)=\delta(x-y) \tag{34.10}$$

and the homogeneous boundary conditions

$$(g_x-k_1g)\Big|_{x=0}=(g_x-k_2g)\Big|_{x=l}=0. \tag{34.11}$$

Statement 34.1. *The Green's function of the mixed problem* (34.3) *can be represented in the form*

$$g(x,y,t)=\sum_{n=1}^{\infty}\frac{2}{l}\sin\frac{\pi nx}{l}\sin\frac{\pi ny}{l}\exp\Big\{-\Big(\frac{\pi na}{l}\Big)^2t\Big\}. \tag{34.12}$$

Substitute the coefficients α_n (34.8) in the explicit form into (34.6) and interchange the summation and the integration to get

$$u(x,t)=\int\limits_0^l g(x,y,t)\varphi(y)dy, \tag{34.13}$$

where $g(x,y,t)$ is defined by (34.12). Hence, the function $g(x,y,t)$ is a solution of the heat equation, for which, by construction, the boundary conditions (34.3) are fulfilled. For $t=0$, we obtain

$$g(x,y,0)=\sum_{n=1}^{\infty}\frac{2}{l}\sin\frac{\pi nx}{l}\sin\frac{\pi ny}{l}=\delta(x-y).$$

Thus, the statement is proved.

◊ The properties of the function $g(x,y,t)$ (34.12) are completely similar to the properties of the Green's function of the Cauchy problem for the heat equation.

34.2 The nonhomogeneous mixed problem

Now we consider the mixed problem for the nonhomogeneous heat equation with nonhomogeneous boundary conditions of the first kind

$$u_t = a^2 u_{xx} + f(x,t), \quad t > 0, \quad 0 < x < l; \tag{34.14}$$
$$u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \quad u(x,0) = \varphi(x). \tag{34.15}$$

Introduce a new function $v(x,t)$ by the relation

$$u(x,t) = w(x,t) + v(x,t),$$

where $w(x,t)$ is an arbitrary function satisfying the conditions $w(0,t) = \mu_1(t)$, $w(l,t) = \mu_2(t)$, for instance,

$$w(x,t) = \mu_1(t) + \frac{x}{l}[\mu_2(t) - \mu_1(t)].$$

Then the function $v(x,t)$ will be found as a solution of the homogeneous boundary value problem

$$v_t = a^2 v_{xx} + \bar{f}(x,t), \qquad t > 0, \qquad 0 < x < l; \tag{34.16}$$
$$v(0,t) = 0, \qquad v(l,t) = 0, \qquad v(x,0) = \bar{\varphi}(x), \tag{34.17}$$
$$\bar{\varphi}(x) = \varphi(x) - w(x,0), \qquad \bar{f}(x,t) = f(x,t) - [w_t - a^2 w_{xx}].$$

As for the one-dimensional d'Alembert equation, the solution of the problem (34.16), (34.17) can be represented as the sum of solutions of two simpler problems: $v(x,t) = L(x,t) + M(x,t)$, where

(a) $L(x,t)$ is a solution of a homogeneous equation with nonzero initial conditions

$$L_t = a^2 L_{xx}, \qquad t > 0, \qquad 0 < x < l; \tag{34.18}$$
$$L(0,t) = 0, \qquad L(l,t) = 0, \qquad L(x,0) = \bar{\varphi}(x). \tag{34.19}$$

According to (34.6), the solution of the problem (34.18), (34.19) has the form

$$L(x,t) = \sum_{n=1}^{\infty} \alpha_n \exp\Big[-\Big(\frac{\pi n a}{l}\Big)^2 t\Big] \sin\frac{\pi n}{l}x,$$

where

$$\alpha_n = \frac{2}{l}\int\limits_0^l \bar{\varphi}(y)\sin\Big(\frac{\pi n}{l}y\Big)dy. \tag{34.20}$$

(b) $M(x,t)$ is a solution of a nonhomogeneous equation with nonzero boundary conditions

$$M_t = a^2 M_{xx} + \bar{f}(x,t), \qquad t > 0, \qquad 0 < x < l; \tag{34.21}$$
$$M(0,t) = 0, \qquad M(l,t) = 0, \qquad M(x,0) = 0. \tag{34.22}$$

The solution of the problem (34.21), (34.22) is sought, as for d'Alembert's equation, in the form

$$M(\vec{x},t) = \sum_{l=1}^{\infty} S_n(t)\sin\frac{\pi n x}{l}. \tag{34.23}$$

Substitute (34.23) into (34.21) and expand the function $\bar f(x,t)$ in a Fourier series:

$$\bar f(x,t)=\sum_{n=1}^{\infty}\bar f_n(t)\sin\frac{\pi nx}{l}, \tag{34.24}$$

$$\bar f_n(\tau)=\frac{2}{l}\int\limits_0^l \bar f(y,\tau)\sin\Big(\frac{\pi n}{l}y\Big)dy.$$

As a result, we obtain the following equation for the function $S_n(t)$:

$$S_n'+\Big(\frac{\pi na}{l}\Big)^2 S_n=\bar f_n(t),\qquad S_n(0)=0. \tag{34.25}$$

The solution of Eq. (34.25) has the form

$$S_n(t)=\int\limits_0^t \bar f_n(\tau)\exp\Big\{\Big(\frac{\pi na}{l}\Big)^2(\tau-t)\Big\}d\tau. \tag{34.26}$$

Hence,

$$M(x,t)=\sum_{n=1}^{\infty}\int\limits_0^t \bar f_n(\tau)\exp\Big[\Big(\frac{\pi na}{l}\Big)^2(\tau-t)\Big]d\tau\,\sin\frac{\pi n}{l}x, \tag{34.27}$$

Substituting the Fourier coefficients $f_n(\tau)$ (34.23) in the explicit form into (34.26) and interchanging the summation and the integration, we get

$$M(x,t)=\int\limits_0^t d\tau\int\limits_0^l g(x,y,t-\tau)\bar f(y,\tau)dy, \tag{34.28}$$

where $g(x,y,t)$ is defined by relation (34.13).

♦ Formula (34.28) is called Duhamel's formula.

Example 34.1 (Problem of the safety of a nuclear reactor). Determine the parameters of a nuclear reactor for which a chain reaction will not occur at any initial distribution of neutrons in the reactor (the reactor will not "explode").

Solution. We already mentioned that in the diffusion approximation, the propagation of neutrons in a reactor is described by the equation

$$u_t=D\Delta u+\gamma u, \tag{34.29}$$

where $u(\vec x,t)$ is the concentration of neutrons, D is the effective coefficient of their diffusion, and γ is their multiplication factor. The coefficients D and γ are determined experimentally.

We assume that the working body of the reactor has the form of a solid sphere of radius b and the initial distribution of neutrons is described by a spherically symmetric function, i.e.,

$$u|_{t=0}=\varphi(r),\qquad r=\sqrt{x^2+y^2+z^2}.$$

We also assume that the neutron concentration at the sphere surface is equal to zero. As a result, we arrive at the following problem:

$$\begin{cases} u_t = D\Delta u + \gamma u, & t > 0, \quad r < b, \\ u|_{t=0} = \varphi(r), & u|_{r=b} = 0. \end{cases} \tag{34.30}$$

If the expression for the neutron concentration $u(\vec{x}, t)$ involves terms increasing exponentially with t, the chain reaction will take place and the reactor will "explode". Therefore, the parameters of the reactor must ensure the fulfillment of the condition

$$|u(\vec{x}, t)| < \infty, \qquad t > 0,$$

regardless of the choice of the initial data — the function $\varphi(r)$.

By virtue of the spherical symmetry of the problem (34.30), we seek its solution in the form

$$u(\vec{r}, t) = u(r, t) = v(r, t)e^{\gamma t}. \tag{34.31}$$

As a result, we arrive at the following problem for the function $v(r, t)$ (see Sec. "Spherical waves"):

$$\begin{gathered} \frac{1}{D} v_t = \frac{1}{r}(rv)_{rr}, \qquad t > 0, \\ v|_{t=0} = \varphi(r), \qquad v|_{r=b} = 0, \quad r < b. \end{gathered} \tag{34.32}$$

Seek a solution of this equation in the form $rv(r, t) = w(r, t)$. Then we have for the function $w(r, t)$ the following problem:

$$\begin{gathered} \frac{1}{D} w_t = w_{rr}, \qquad t > 0, \quad r < b, \\ w|_{t=0} = r\varphi(r), \qquad w|_{r=0} = w|_{r=b} = 0. \end{gathered} \tag{34.33}$$

The solution of this problem was obtained earlier [see (34.14)] and it has the form

$$w(r, t) = \sum_{n=1}^{\infty} \alpha_n e^{-(\pi n/b)^2 D t} \sin \frac{\pi n r}{b}, \tag{34.34}$$

where

$$\alpha_n = \frac{2}{b} \int_0^b r\varphi(r) \sin \frac{\pi n r}{b} dr. \tag{34.35}$$

Returning to the original notations, we get

$$u(r, t) = \sum_{n=1}^{\infty} \alpha_n \frac{1}{r} \sin \frac{\pi n r}{b} \exp\Big\{\Big[\gamma - \Big(\frac{\pi n}{b}\Big)^2 D\Big] t\Big\}.$$

Thus, the answer to the problem's question is determined by the inequality

$$\gamma - \frac{\pi^2 D}{b^2} < 0$$

that should be satisfied by the reactor parameters. Hence, for $b < b_{\rm cr}$ and given D and γ, the chain reaction will not occur. The critical radius is defined by the relation

$$b_{\rm cr} = \pi\sqrt{\frac{D}{\gamma}}. \tag{34.36}$$

This radius is associated with the critical mass

$$m_{\rm cr} = \frac{4}{3}\pi\rho b_{\rm cr}^3.$$

Example 34.2. In an infinitely long homogeneous cylinder of radius b, starting from the time $t = 0$, heat sources of constant density Q are operative. The initial temperature of the cylinder is T. Find the temperature distribution in the cylinder at $t > 0$ if the cylinder surface is kept at constant temperature Θ.

Solution. The mathematical formulation of the problem is

$$u_t = a^2\Delta u + Q; \qquad u|_{t=0} = T, \qquad u|_{r=b} = \Theta. \tag{34.37}$$

Write Eq. (34.37) in cylindrical coordinates. Since the initial and boundary conditions are independent of z and φ, then

$$u(r, \varphi, z, t) = u(r, t).$$

Thus, for the function $u(r, t)$ we get

$$\frac{\partial u}{\partial t} = a^2\Big(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\Big) + Q.$$

Reduce this problem to a mixed problem with homogeneous boundary conditions:

$$u(r, t) = v(r, t) + \Theta.$$

The function $v(r, t)$ is a solution of the mixed problem

$$\frac{\partial v}{\partial t} = a^2\Big(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\Big) + Q, \quad v|_{t=0} = T - \Theta, \quad v|_{r=b} = 0.$$

The change

$$v(r, t) = e^{Qt}w(r, t)$$

yields for the function $w(r, t)$ the equation

$$\frac{\partial w}{\partial t} = a^2\Big(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\Big) \tag{34.38}$$

with the initial condition $w|_{t=0} = T - \Theta$ and the boundary condition $w|_{r=b} = 0$.

Seek a solution of Eq. (34.38) by separation of variables in the form

$$w(r, t) = R(r)T(t).$$

Separating the variables, we have

$$\frac{T'}{a^2T} = \frac{R''(r) + R'(r)/r}{R} = \lambda,$$

whence for the function $T(t)$ we get

$$T' - \lambda a^2 T = 0; \qquad |T(t)| < \infty, \quad t > 0, \tag{34.39}$$

and the function $R(r)$ is a solution of the Sturm–Liouville problem

$$R'' + \frac{1}{r}R' - \lambda R = 0, \qquad |R(r)| < \infty, \quad R(b) = 0, \quad r < b. \tag{34.40}$$

The problem (34.40) was considered in Sec. "The Sturm–Liouville problem for Bessel's equation" of Part III, where we have obtained

$$R_n(r) = C_n J_0\Big(\alpha_n^0 \frac{r}{b}\Big), \qquad \lambda_n = -\Big(\frac{\alpha_n^0}{b}\Big)^2, \quad n = \overline{1,\infty},$$

where α_n^0 are roots of Bessel's function $J_0(x)$. Then from Eq. (34.39) we find

$$T_n(t) = B_n e^{\lambda_n a^2 t} = B_n e^{-(\alpha_n^0 a/b)^2 t}, \qquad n = \overline{1,\infty},$$

and the solution of the problem (34.38) will be written in the form

$$w(x,t) = \sum_{n=1}^{\infty} A_n e^{-(\alpha_n^0 a/b)^2 t} J_0\Big(\alpha_n^0 \frac{r}{b}\Big),$$

where $A_n = B_n C_n$. To determine the coefficients A_n, we use the initial condition

$$w(x,0) = \sum_{n=1}^{\infty} A_n J_0\Big(\alpha_n^0 \frac{r}{b}\Big) = T - \Theta.$$

Expanding the function $T - \Theta$ on the interval $]0, b[$ in a Fourier–Bessel series in orthogonal functions $J_0(\alpha_n^0 r/b)$, $n = \overline{1,\infty}$, we obtain (see Example III.11.1)

$$A_n = (T - \Theta)\int\limits_0^b \frac{r J_0(\alpha_n^0 r/b)}{\|J_0(\alpha_n^0 r/b)\|^2} dr = \frac{2(T-\Theta)}{\alpha_n^0 J_1(\alpha_n^0)},$$

and the solution of the original problem (34.37), in view of (34.38) and (34.39), becomes

$$u(r,t) = \Theta + \sum_{n=1}^{\infty} \frac{2(T-\Theta)}{\alpha_n^0 J_1(\alpha_n^0)} J_0\Big(\alpha_n^0 \frac{r}{b}\Big) e^{-(a\alpha_n^0/b)^2 t + Qt}.$$

Example 34.3. Solve the mixed problem

$$u_t = u_{xx}, \qquad u_x|_{x=0} = 1, \quad u|_{x=1} = 0, \quad u|_{t=0} = 0. \tag{34.41}$$

Solution. Seek a solution of the problem (34.41) in the form

$$u(x,t) = x - 1 + v(x,t). \tag{34.42}$$

Then for the function $v(x,t)$ we obtain a mixed problem with homogeneous boundary conditions:

$$v_t = v_{xx}, \qquad v_x|_{x=0} = 0, \quad v|_{x=1} = 0, \quad v|_{t=0} = 1 - x. \tag{34.43}$$

Seek a solution of the problem (34.43) by separation of variables in the form

$$v(x,t) = X(x)T(t). \tag{34.44}$$

For $T(t)$, we have the equation

$$\dot{T} = \lambda T, \tag{34.45}$$

and the function $X(x)$ is a solution of the Sturm–Liouville problem

$$X''(x) = \lambda X(x), \qquad X'(0) = X'(1) = 0 \tag{34.46}$$

with eigenfunctions

$$X_n(x) = A_n \cos\frac{\pi(1+2n)x}{2}, \quad n = \overline{0,\infty} \tag{34.47}$$

and eigenvalues (see Example III.2.3)

$$\lambda_n = -\Big[\frac{\pi(1+2n)}{2}\Big]^2, \quad n = \overline{0,\infty}.$$

Substitution of (34.47) into (34.45) yields

$$T_n(t) = B_n \exp\Big\{-\Big[\frac{\pi(1+2n)}{2}\Big]^2 t\Big\}.$$

Then

$$v_n(x,t) = \sum_{n=0}^{\infty} \bar{A}_n \exp\Big\{-\Big[\frac{\pi(1+2n)}{2}\Big]^2 t\Big\} \cos\frac{\pi(1+2n)x}{2}. \tag{34.48}$$

Here $\bar{A}_n = A_n B_n$. From the boundary condition (34.42) we find

$$\sum_{n=1}^{\infty} \bar{A}_n \cos\frac{\pi(1+2n)x}{2} = x - 1.$$

Expanding the right side in a Fourier series in functions $X_n(x)$ (34.47) and equating the coefficients of identical functions $X_n(x)$, we get

$$\bar{A}_n = 2\int_0^1 (x-1)\cos\frac{\pi(1+2n)x}{2}dx = \frac{8}{\pi^2(1+2n)^2}. \tag{34.49}$$

Substituting (34.49) and (34.48) into (34.42), we obtain the solution of the problem (34.41)

$$u(x,t) = x - 1 + \\ +\frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(1+2n)^2}\exp\Big\{-\Big[\frac{\pi(1+2n)}{2}\Big]^2 t\Big\}\cos\frac{\pi(1+2n)x}{2}.$$

Example 34.4. Solve the mixed problem

$$\begin{gathered} u_t - u_{xx} - 9u = 4\sin^2 t\cos 3x - 9x^2 - 2, \\ u_x|_{x=0} = 0, \quad u_x|_{x=\pi} = 2\pi, \quad u|_{t=0} = x^2 + 2. \end{gathered} \tag{34.50}$$

Solution. Seek a solution of the problem (34.50) in the form

$$u(x,t) = x^2 + v(x,t). \tag{34.51}$$

Then the function $v(x,t)$ is a solution of the mixed problem with homogeneous boundary conditions

$$\begin{gathered} v_t - v_{xx} - 9v = 4\sin^2 t\cos 3x, \\ v_x|_{x=0} = 0, \quad v_x|_{x=\pi} = 0, \quad v|_{t=0} = 2. \end{gathered} \tag{34.52}$$

Let us reduce the problem (34.52), putting

$$v(x,t) = w(x,t) + \bar{w}(x,t).$$

Here the function $w(x,t)$ is a solution of the mixed problem for the homogeneous heat equation

$$w_t - w_{xx} - 9w = 0, \quad w_x|_{x=0} = w_x|_{x=\pi} = 0, \; w|_{t=0} = 2, \tag{34.53}$$

and the function $\bar{w}$ is a solution of the nonhomogeneous equation with zero initial conditions

$$\begin{gathered} \bar{w}_t - \bar{w}_{xx} - 9\bar{w} = 4\sin^2 t\cos 3x, \\ \bar{w}_x|_{x=0} = \bar{w}_x|_{x=\pi} = 0, \quad \bar{w}|_{t=0} = 0. \end{gathered} \tag{34.54}$$

The solution of the problem (34.53) has the form

$$w(x,t) = \sum_{n=0}^{\infty} \bar{A}_n e^{(9-n^2)t}\cos nx.$$

From the initial condition, we find

$$w(x,0) = \sum_{n=0}^{\infty} \bar{A}_n \cos nx = 2.$$

Hence, $\bar{A}_0 = 2$, $\bar{A}_n = 0$, $n = \overline{1,\infty}$, and

$$w(x,t) = 2e^{9t}. \tag{34.55}$$

The solution of the problem (34.54) is sought in the form

$$\bar{w}(x,t) = \sum_{n=0}^{\infty} S_n \cos nx. \tag{34.56}$$

Substituting (34.56) into (34.54) and expanding the right side into a Fourier series in $X_n(x) = \cos nx$

$$4\sin^2 t\cos 3x = 4\sin^2 t\sum_{n=0}^{\infty} \delta_{n3}\cos nx,$$

we get

$$\sum_{n=0}^{\infty} S_n'(t)\cos nx = 4\sin^2 t \sum_{n=0}^{\infty} \delta_{n3}\cos nx.$$

Equating the coefficients of identical functions $X_n(x) = \cos nx$, we have for $n = 3$

$$S_3' = 4\sin^2 t, \qquad S_3(0) = 0$$

and for $n \neq 3$

$$S_n' + (n^2 - 9)S_n = 0, \qquad S_n(0) = 0.$$

Hence,

$$S_n(t) = (2t - \sin 2t)\delta_{n3}, \qquad n = \overline{0, \infty}.$$

Finally,

$$u(x,t) = x^2 + 2e^{9t} + (2t - \sin 2t)\cos 3x.$$

35 The Cauchy problem for a multidimensional heat equation. Green's function of the Cauchy problem

♦ By the Green's function of the heat equation

$$u_t = a^2\Delta u \tag{35.1}$$

is meant a generalized function $G(\vec{x}, \vec{y}, t)$, $\vec{x}, \vec{y} \in \mathbb{R}^n$, which satisfies this equation and the initial condition

$$G(\vec{x}, \vec{y}, 0) = \delta(\vec{x} - \vec{y}). \tag{35.2}$$

Statement 35.1. *The Green's function of the heat equation*

$$\frac{\partial G}{\partial t} = a^2\Delta G, \qquad G(\vec{x}, \vec{y}, 0) = \delta(\vec{x} - \vec{y}) \tag{35.3}$$

has the form

$$\begin{aligned} G(\vec{x}, \vec{y}, t) = G(x_1, y_1, t)G(x_2, y_2, t)\cdots G(x_n, y_n, t) = \\ = \prod_{k=1}^{n} G(x_k, y_k, t), \end{aligned} \tag{35.4}$$

where $G(x_k, y_k, t)$, $k = \overline{1, n}$ is the Green's function of the Cauchy problem for the one-dimensional heat equation (33.13).

Find the partial derivatives

$$\frac{\partial G(\vec{x}, \vec{y}, t)}{\partial t} = \sum_{k=1}^{n} \frac{G(\vec{x}, \vec{y}, t)}{G(x_k, y_k, t)} \frac{\partial G(x_k, y_k, t)}{\partial t},$$

$$\Delta G(\vec{x}, \vec{y}, t) = \sum_{k=1}^{n} \frac{G(\vec{x}, \vec{y}, t)}{G(x_k, y_k, t)} \frac{\partial^2 G(x_k, y_k, t)}{\partial x_k^2}.$$

At the same time,

$$\frac{\partial G(x_k, y_k, t)}{\partial t} = a^2 \frac{\partial G(x_k, y_k, t)}{\partial x_k^2}, \qquad k = \overline{1, n}.$$

Thus, $G(\vec{x}, \vec{y}, t)$ satisfies Eq. (35.3). For $t = 0$ we get

$$G|_{t=0} = \prod_{k=1}^{n} G(x_k, y_k, t)\Big|_{t=0} = \prod_{k=1}^{n} \delta(x_k - y_k) = \delta(\vec{x} - \vec{y}),$$

which was to be shown.

Statement 35.2. *The solution of the Cauchy problem for the homogeneous heat equation*

$$\frac{\partial u}{\partial t} = a^2 \Delta u, \qquad u|_{t=0} = \varphi(\vec{x}), \qquad x \in \mathbb{R}^n, \tag{35.5}$$

has the form

$$u(\vec{x}, t) = \int\limits_{\mathbb{R}^3} G(\vec{x}, \vec{y}, t)\varphi(\vec{y})d\vec{y}. \tag{35.6}$$

Obviously, the function (35.6) satisfies Eq. (35.5). In addition,

$$u(\vec{x}, 0) = \int\limits_{\mathbb{R}^n} G(\vec{x}, \vec{y}, 0)\varphi(\vec{y})d\vec{y} = \int\limits_{\mathbb{R}^n} \delta(\vec{x} - \vec{y})\varphi(\vec{y})d\vec{y} = \varphi(\vec{x}),$$

which was to be demonstrated.

◊ From the explicit form of Green's function (33.13), one can easily obtain

$$G(\vec{x}, \vec{y}, t) = \Big(\frac{1}{\sqrt{4\pi a^2 t}}\Big)^n \exp\Big[-\frac{(\vec{x} - \vec{y})^2}{4a^2 t}\Big]. \tag{35.7}$$

Substitution of (35.7) into (35.5) yields

$$u(\vec{x}, t) = \Big(\frac{1}{\sqrt{4\pi a^2 t}}\Big)^n \int\limits_{\mathbb{R}^n} \varphi(\vec{y}) \exp\Big[-\frac{(\vec{x} - \vec{y})^2}{4a^2 t}\Big]d\vec{y}. \tag{35.8}$$

♦ Relation (35.8) is called Poisson's formula of the Cauchy problem for the heat equation.

Statement 35.3. *The solution of the Cauchy problem for the nonhomogeneous heat equation*

$$\frac{\partial u}{\partial t} = a^2 \Delta u + f(\vec{x}, t), \qquad u|_{t=0} = 0, \tag{35.9}$$

has the form

$$u(\vec{x}, t) = \int\limits_0^t d\tau \int\limits_{\mathbb{R}^3} G(\vec{x}, \vec{y}, t - \tau) f(\vec{y}, \tau)d\vec{y}. \tag{35.10}$$

♦ *Relation* (35.10) *is called Duhamel's formula.*

The proof is similar to that for the one-dimensional case.

Corollary. The solution of the Cauchy problem for the nonhomogeneous heat equation

$$\frac{\partial u}{\partial t} = a^2\Delta u + f(\vec{x}, t), \quad u|_{t=0} = \varphi(\vec{x}), \quad \vec{x} \in \mathbb{R}^n, \quad t > 0,$$

is the sum of solutions (35.6) and (35.10) and has the form

$$u(\vec{x}, t) = \Big(\frac{1}{\sqrt{4\pi a^2 t}}\Big)^n \int\limits_{\mathbb{R}^n} \varphi(\vec{y}) \exp\Big[-\frac{(\vec{x}-\vec{y})^2}{4a^2 t}\Big] d\vec{y} +$$
$$+\int\limits_0^t d\tau \Big(\frac{1}{\sqrt{4\pi a^2(t-\tau)}}\Big)^n \int\limits_{\mathbb{R}^n} f(\vec{y}, \tau) \exp\Big[-\frac{(\vec{x}-\vec{y})^2}{4a^2(t-\tau)}\Big] d\vec{y}. \tag{35.11}$$

Proof is obvious.

Example 35.1. Find a solution of the Cauchy problem

$$u_t = 2u_{xx}, \qquad u\Big|_{t=0} = \exp(-4x^2 + 8x).$$

Solution. According to Poisson's formula (35.8), the solution of the Cauchy problem for the heat equation with $a = \sqrt{2}$ has the form

$$u(x, t) = \frac{1}{2\sqrt{2\pi t}} \int\limits_{-\infty}^{\infty} \exp(-4y^2 + 8y) \exp\Big[-\frac{(x-y)^2}{8t}\Big] dy. \tag{35.12}$$

Complete the perfect square in the exponential power of the integrand

$$-4y^2 + 8y - \frac{(x-y)^2}{8t} = -\frac{1}{8t}[32y^2 t - 64yt + x^2 - 2xy + y^2] =$$
$$= -\frac{1}{8t}\Big\{(32t+1)\Big[y^2 - 2\frac{32t+x}{32t+1}y\Big] + x^2\Big\} =$$
$$= -\frac{1}{8t}\Big[(32t+1)\Big(y - \frac{32t+x}{32t+1}\Big)^2 - \frac{(32t+x)^2}{32t+1} + x^2\Big].$$

Making in (35.12) the change

$$y = z + \frac{32t+x}{32t+1},$$

we find

$$u(x, t) = \frac{1}{2\sqrt{2\pi t}} \exp\Big\{-\frac{1}{8t}\Big[x^2 - \frac{(32t+x)^2}{32t+1}\Big]\Big\} \int\limits_{-\infty}^{\infty} \exp\Big(-\frac{32t+1}{8t}z^2\Big) dz.$$

In view of the value of the Gauss integral

$$\int\limits_{-\infty}^{\infty} e^{-a^2 z^2} dz = \frac{\sqrt{\pi}}{a},$$

we finally obtain

$$u(x,t) = \frac{1}{\sqrt{32t+1}} \exp\Big(\frac{4x^2 - 8x - 128t}{32t+1}\Big).$$

Example 35.2. Solve the Cauchy problem

$$u_t = a^2 u_{xx} + f(x,t), \qquad u\big|_{t=0} = \varphi(x), \quad x \in \mathbb{R}, \quad t > 0, \tag{35.13}$$

for

$$a^2 = 1, \qquad f(x,t) = e^{-t}\cos x, \qquad \varphi(x) = \cos x. \tag{35.14}$$

Solution. The solution of the problem (35.13), (35.14) is given by Poisson's formula (35.8) with $n = 1$:

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4a^2 t} \cos y \, dy +$$
$$+ \int_0^t d\tau \int_{-\infty}^{\infty} \frac{1}{2a\sqrt{\pi(t-\tau)}} e^{(x-y)^2/4a^2(t-\tau)} e^{-\tau} \cos y \, dy. \tag{35.15}$$

The first integral in (35.15)

$$I_1 = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4a^2 t} \cos y \, dy,$$

by the change of variables

$$\frac{y-x}{2a\sqrt{t}} = z, \qquad y = 2a\sqrt{t}z + x, \qquad dy = 2a\sqrt{t}\, dz, \tag{35.16}$$

is reduced to an integral of the form

$$I_1 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \big[\cos 2a\sqrt{t}z \cos x + \sin 2a\sqrt{t}z \sin x\big] e^{-z^2} dz, \tag{35.17}$$

calculated in Sec. "Green's functions of the Cauchy problem" when proving Property 3:

$$\int_{-\infty}^{\infty} e^{-z^2} \cos \varkappa z \, dz = \sqrt{\pi} e^{-\varkappa^2/4}, \tag{35.18}$$

$$\int_{-\infty}^{\infty} e^{-z^2} \sin \varkappa z \, dz = 0. \tag{35.19}$$

Substitution of (35.18) and (35.19) into (35.17) yields

$$I_1 = \cos x e^{-a^2 t}. \tag{35.20}$$

Successive calculation of the integrals with $a = 1$:

$$\int_{-\infty}^{\infty} e^{-(y-x)^2/4a^2(t-\tau)} e^{-\tau} \cos y \, dy \Big|_{a=1} = 2a\sqrt{\pi(t-\tau)} e^{-a^2(t-\tau)} \cos x \Big|_{a=1} =$$
$$= 2\sqrt{\pi(t-\tau)} e^{-(t-\tau)} \cos x,$$
$$\int_0^t \cos x \, e^{-\tau-(t-\tau)} d\tau = \cos x \, e^{-t} \int_0^t d\tau = e^{-t} t \cos x$$

yields for the second integral in (35.15)

$$I_2 = \int_0^t d\tau \int_{-\infty}^{\infty} \frac{1}{2a\sqrt{\pi(t-\tau)}} e^{(x-y)^2/4a^2(t-\tau)} e^{-\tau} \cos y \, dy \Big|_{a=1} = e^{-t} t \cos x. \qquad (35.21)$$

Substitution of (35.20) and (35.21) into (35.15) yields

$$u(x,t) = e^{-t}(1+t)\cos x.$$

Example 35.3. Let there be an infinitely long homogeneous rod and through its surface there occurs convective heat exchange with a medium of zero temperature. Find the temperature distribution in the rod if the initial temperature distribution has the form $u|_{t=0} = 2\sqrt{\pi}\delta(x)$.

Solution. The mathematical formulation of the problem is (see Eq. (11.5))

$$u_t = a^2 u_{xx} - u, \qquad u|_{t=0} = 2\sqrt{\pi}\delta(x).$$

Seek a solution of the problem in the form

$$u(x,t) = e^{-t} v(x,t).$$

Then for the function $v(x,t)$ we have the Cauchy problem

$$v_t = a^2 v_{xx}, \qquad v|_{t=0} = 2\sqrt{\pi}\delta(x),$$

whose solution is given by Poisson's formula (35.8) with $n = 1$. In view of the properties of the delta-function, we obtain

$$u(x,t) = \frac{\exp(-t)}{\sqrt{a^2 t}} \exp\Big[-\frac{x^2}{4a^2 t}\Big]. \qquad (35.22)$$

The plot of the function (35.22) for $a = 1$ is given in Fig. 40. Here we have used for the delta-function the delta-like sequence

$$\delta(x,\varepsilon) = \begin{cases} 1/2\varepsilon & |x| \leqslant \varepsilon, \\ 0 & |x| > \varepsilon, \end{cases} \qquad \varepsilon > 0$$

with $\varepsilon = 0.01$.

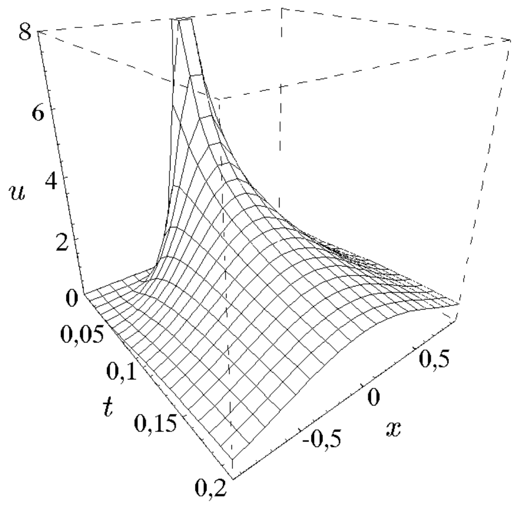

Fig. 40.

Example 35.4. Solve the Cauchy problem

$$u_t = a^2\Delta u + f(\vec{x}, t), \qquad u\Big|_{t=0} = \varphi(\vec{x}), \quad \vec{x} \in \mathbb{R}^2, \tag{35.23}$$

for

$$a = 1, \qquad f(\vec{x}, t) = \sin x_1 \sin x_2 \sin t, \qquad \varphi(\vec{x}) = 1. \tag{35.24}$$

Solution. According to Poisson's formula (35.8), for $n = 2$, in view of (35.24), we find

$$\begin{aligned} u(\vec{x}, t) = &\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{1}{(2\sqrt{\pi t})^2} e^{-[(y_1-x_1)^2+(y_2-x_2)^2]/4t} dy_1 dy_2 + \\ &+ \int\limits_0^t d\tau \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{1}{(2\sqrt{\pi(t-\tau)})^2} e^{-[(y_1-x_1)^2+(y_2-x_2)^2]/[4(t-\tau)]} \times \\ &\times \sin y_1 \sin y_2 \sin\tau \, dy_1 dy_2. \end{aligned} \tag{35.25}$$

The change of variables

$$\frac{y_1 - x_1}{2\sqrt{t}} = u, \quad \frac{y_2 - x_2}{2\sqrt{t}} = v, \qquad \frac{y_1 - x_1}{2\sqrt{t-\tau}} = \bar{u}, \quad \frac{y_2 - x_2}{2\sqrt{t-\tau}} = \bar{v}$$

yields

$$u(\vec{x}, t) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} e^{-u^2} du \int\limits_{-\infty}^{\infty} e^{-v^2} dv +$$

$$+\frac{1}{\pi}\int\limits_0^t d\tau \, \sin\tau \int\limits_{-\infty}^{\infty} e^{-\bar{u}^2} \sin[2\sqrt{t-\tau}\bar{u} + x] d\bar{u} \int\limits_{-\infty}^{\infty} e^{-\bar{v}^2} \sin[2\sqrt{t-\tau}\bar{v} + x] d\bar{v}, \tag{35.26}$$

whence, in view of the relation

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

and formulas (35.18), (35.19), we obtain

$$\begin{aligned} u(\vec{x}, t) &= 1 + \sin x_1 \sin x_2 \int\limits_0^t \sin\tau e^{-2(t-\tau)} d\tau = 1 + \sin x_1 \sin x_2 e^{-2t} \int\limits_0^t \sin\tau e^{2\tau} d\tau = \\ &= 1 + \sin x_1 \sin x_2 e^{-2t} \frac{1}{5}\Big[e^{2\tau}(2\sin\tau - \cos 2\tau]\Big|_0^t = \\ &= 1 + \frac{1}{5}\sin x_1 \sin x_2 e^{-2t}\big[(2\sin t - \cos 2t + e^{-2t}\big]. \end{aligned}$$

Example 35.5. Solve the Cauchy problem

$$u_t = a^2\Delta u + f(\vec{x}, t), \qquad u\Big|_{t=0} = \varphi(\vec{x}), \quad \vec{x} \in \mathbb{R}^3, \tag{35.27}$$

for

$$a^2 = 2, \qquad f(\vec{x}, t) = t\cos x_1, \qquad \varphi(\vec{x}) = \cos x_2 \cos x_3. \tag{35.28}$$

Solution. According to Poisson's formula (35.8) for $n = 3$ (see the preceding example), in view of (35.28), we have

$$u(\vec{x}, t) = I_1 + \int_0^t I_2 \tau \, d\tau, \tag{35.29}$$

where

$$I_1 = \int_{-\infty}^{\infty} \frac{1}{2a\sqrt{\pi t}} e^{-(x_1-y_1)^2/4a^2t} dy_1 \int_{-\infty}^{\infty} \frac{1}{2a\sqrt{\pi t}} e^{-(x_2-y_2)^2/4a^2t} \cos y_2 \, dy_2 \times$$
$$\times \int_{-\infty}^{\infty} \frac{1}{2a\sqrt{\pi t}} e^{-(x_3-y_3)^2/4a^2t} \cos y_3 \, dy_3 \tag{35.30}$$

and

$$I_2 = \int_{-\infty}^{\infty} \frac{1}{2a\sqrt{\pi(t-\tau)}} e^{-(x_1-y_1)^2/4a^2(t-\tau)} dy_1 \int_{-\infty}^{\infty} \frac{1}{2a\sqrt{\pi(t-\tau)}} e^{-(x_2-y_2)^2/4a^2(t-\tau)} \cos y_2 \, dy_2 \times$$
$$\times \int_{-\infty}^{\infty} \frac{1}{2a\sqrt{\pi(t-\tau)}} e^{-(x_3-y_3)^2/4a^2(t-\tau)} \cos y_3 \, dy_3. \tag{35.31}$$

All integrals involved in (35.30) and (35.31) were calculated in two preceding examples; therefore, the quantities I_1 and I_2 can be written at once:

$$I_1 = \cos x_2 \cos x_3 e^{-2a^2 t}, \tag{35.32}$$
$$I_2 = \cos x_1 e^{-a^2(t-\tau)}. \tag{35.33}$$

Integration of I_2 with respect to τ yields

$$\int_0^t I_2 \tau \, d\tau = \int_0^t \cos x_1 e^{-a^2(t-\tau)} \tau \, d\tau =$$
$$= \cos x_1 e^{-a^2 t} \int_0^t e^{a^2\tau} \tau \, d\tau = \frac{\cos x_1}{a^4} \big[a^2 t - 1 + e^{-a^2 t}\big]. \tag{35.34}$$

Substituting (35.32) and (35.34) into (35.29), we find

$$u(\vec{x}, t) = \cos x_2 \cos x_3 e^{-2a^2 t} + \frac{\cos x_1}{a^4} \big[a^2 t - 1 + e^{-a^2 t}\big],$$

and, in view of the fact that $a^2 = 2$, this yields

$$u(\vec{x}, t) = \cos x_2 \cos x_3 e^{-4t} + \frac{1}{4} \cos x_1 \big[2t - 1 + e^{-2t}\big].$$

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Educational Edition

Национальный исследовательский
Томский политехнический университет

БАГРОВ Владислав Гаврилович
БЕЛОВ Владимир Владимирович
ЗАДОРОЖНЫЙ Валерий Николаевич
ТРИФОНОВ Андрей Юрьевич

МЕТОДЫ МАТЕМАТИЧЕСКОЙ ФИЗИКИ

Специальные функции. Уравнения математической физики

Учебное пособие

Издательство Томского политехнического университета, 2012

На английском языке

Published in author's version

Typesetting *V.N. Romanenko*
Cover design *A.I. Sidorenko*

Printed in the TPU Publishing House in full accordance with the quality of the given make up page

Signed for the press 24.12.2012. Format 60x84/8. Paper "Snegurochka".
Print XEROX. Arbitrary printer's sheet 29.89. Publisher's signature 27.04.
Order 1510-12. Size of print run 100.

Tomsk Polytechnic University
Quality management system
of Tomsk Polytechnic University was certified by
NATIONAL QUALITY ASSURANCE on BS EN ISO 9001:2008

PUBLISHING HOUSE TPU. 30, Lenina Ave, Tomsk, 634050, Russia
Tel/fax: +7 (3822) 56-35-35, www.tpu.ru